Distribution of Orders of Abelian Groups

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In a recent paper, Gallian [1] gives an account of some computer related projects in group theory. In one of these projects a study was made (see TABLE 1) to determine, for each integer k, the percentages of integers in a given interval which are orders of exactly k Abelian groups. For example, the data in Table 1 shows that in the interval from 50001 to 50500, 19.80% of the integers correspond to orders of two non-isomorphic Abelian groups while 2.00% correspond to orders of six non-isomorphic Abelian groups. One quickly notices that the percentages in TABLE 1 are, to a large extent, independent of the interval of integers analyzed. The purpose of this paper is to explain this occurrence.

No. of groups	1 – 10000	50001 – 50500	500001 – 505000	500001 – 550000	900001 – 901000	999001 – 1000000
1	60.83	61.00	60.82	60.80	60.90	60.80
2	20.08	19.80	20.04	20.06	19.90	19.70
3	7.44	7.20	7.38	7.42	7.70	7.70
4	2.20	2.20	2.22	2.23	2.40	2.30
5	3.21	3.40	3.20	3.20	3.00	3.20
6	1.46	2.00	1.42	1.45	1.20	1.60
7	1.51	1.60	1.50	1.47	1.60	1.40
8	0.08	0.00	0.14	0.10	0.10	0.10
9	0.22	0.20	0.28	0.22	0.30	0.10
10 or more	2.97	2.60	3.00	3.05	2.90	3.10

Percentages of integers in given ranges that are the order of the given number of distinct Abelian groups.

TABLE 1

The number of Abelian groups of a given order can easily be determined by applying the fundamental theorem of finite abelian groups [2, Sec. 214]: if $r = p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots p_s^{e_s}$ where the p_i's are distinct primes, the number of Abelian groups with order r is $g(r) = \alpha(e_1)\alpha(e_2)\alpha(e_3) \cdots \alpha(e_s)$, where $\alpha(e)$ is the number of ways e can be written as the sum of positive integers. Some values of the function α (often called the **partition function**) are given in TABLE 2.

The question posed by Gallian's paper can now be restated in number theoretic terms: For a given integer k, what percent of the integers in a given interval satisfy the equations $g(r) = k$? Letting Z be the set of positive integers, define $A_k = \{r \in Z: g(r) = k\}$. To answer the preceding question we will,

e:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\alpha(e)$:	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	213	297	385	490	627

Number of ways, $\alpha(e)$, of writing the positive integer e as a sum of positive integers.

TABLE 2

for convenience, analyze A_k as a whole rather than over finite intervals. To do this, we introduce the following concept of density. If $X \subset Z$, let $N(X, n)$ be the number of elements in X less than or equal to n. Then define the **density** of X to be $D(X) = \lim_{n\to\infty} N(X, n)/n$ if the limit exists. Now let us consider the density of the sets A_k.

From the definition of $g(r)$, it is clear that $g(r) = 1$ if and only if no prime appears more than once in the prime factorization of r. For $g(r)$ to equal 2, exactly one prime factor of r must be repeated. One can use TABLE 2 to construct in this manner the solution sets for $g(r) = k$ for any k; see TABLE 3. It follows from this table that, for example, $2^3 \cdot 3^2 \cdot 5 \cdot 13$ is an element of A_6 and $3^2 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 23$ is an element of A_8. The structure of A_k may vary from trivial (e.g., A_{13} is empty) to complex. For example, since $30 = \alpha(9) = \alpha(2)\,\alpha(3)\,\alpha(4) = \alpha(2)\,\alpha(7)$, an element is in A_{30} if it has exactly one prime repeated nine times, or if it has exactly one prime appearing twice, another appearing 3 times, and a third appearing four times, or again if it has one prime appearing twice and another seven times.

k	Description of the elements of A_k
1	No prime factor can be repeated.
2	Exactly 1 prime factor must appear twice.
3	Exactly 1 prime factor must appear 3 times.
4	Exactly 2 prime factors must appear twice each.
5	Exactly 1 prime factor must appear 4 times.
6	Exactly 1 prime factor must appear twice and a second must appear 3 times.
7	Exactly 1 prime factor must appear 5 times.
8	Exactly 3 prime factors must appear twice each.
9	Exactly 2 prime factors must appear 3 times each.
10	Exactly 1 prime factor must appear twice and a second must appear 4 times.
11	Exactly one prime factor must appear 6 times.
12	Exactly 2 prime factors must appear twice each and a third must appear three times.
13	A_{13} is empty since 13 is a prime that is not $\alpha(e)$ for any e.
14	Exactly 1 prime factor must appear twice and a second must appear 5 times.

Criteria that an integer be in A_k, the set of integers n such that there exist exactly k Abelian groups of order n.

TABLE 3

Beginning now with A_1, if $P_i = \{r \in Z\colon p_i^2 | r\}$ where p_i is the ith prime, $A_1 = Z - \bigcup_{i\in Z} P_i = \bigcap_{i\in Z}(Z - P_i)$. It is clear that the density of $P_i = D(P_i) = 1/p_i^2$ and that $D(Z - P_i) = 1 - 1/p_i^2$. Intuitively it seems that $D(A_1) = \Pi_{i\in Z}(1 - 1/p_i^2)$ but one needs a couple of lemmas to make this rigorous.

LEMMA 1. *$D(X \cup Y) = D(X) + D(Y) - D(X \cap Y)$ when any three of these terms exist.*

Proof. Clearly, $N(X \cup Y, n) = N(X, n) + N(Y, n) - N(X \cap Y, n)$ so

$$D(X \cup Y) = \lim_{n\to\infty} N(X \cup Y, n)/n$$

$$= \lim_{n\to\infty} N(X, n)/n + \lim_{n\to\infty} N(Y, n)/n - \lim_{n\to\infty} N(X \cap Y, n)/n$$

$$= D(X) + D(Y) - D(X \cap Y).$$

The sets $P_1, P_2, P_3, \ldots$ are in a sense independent in that each P_i is a set of multiples of a p_i^2 and the p_i's are relatively prime. This allows the P_i's to satisfy the hypothesis of the next lemma.

LEMMA 2. *Let $\{X_i\}$ be a sequence of subsets of Z. If the X_i's have the property that for any $m \in Z$*

$$D\left(\bigcup_{j\leqq m} X_j \cap X_{m+1}\right) = D\left(\bigcup_{j\leqq m} X_j\right)\cdot D(X_{m+1}),$$

then for any $m \in Z$

$$D\left(\bigcup_{j\leqq m} X_j\right) = 1 - \Pi_{j\leqq m}\,[1 - D(X_j)].$$

Proof. This is trivially true for $m = 1$. Assuming that

$$D\left(\bigcup_{j\leqq m} X_j\right) = 1 - \Pi_{j\leqq m}\,[1 - D(X_j)],$$

apply Lemma 1 to get

$$D\left(\bigcup_{j\leqq m+1} X_j\right) = D\left(\bigcup_{j\leqq m} X_j\right) + D(X_{m+1}) - D\left[\bigcup_{j\leqq m} X_j \cap X_{m+1}\right]$$

$$= 1 - \Pi_{j\leqq m}\,[1 - D(X_j)] + D(X_{m+1}) - D\left(\bigcup_{j\leqq m} X_j\right)\cdot D(X_{m+1})$$

$$= 1 - \Pi_{j\leqq m+1}[1 - D(X_j)].$$

Thus the lemma is proved by induction.

Another important property of the P_i's is that for each $n \in Z$, $N(P_i, n) = [n/p_i^2] \leqq n/p_i^2$ so $N(P_i, n)/n \leqq 1/p_i^2$ for all n. Thus, since $\Sigma_{i\in Z}\, 1/p_i^2$ converges, this will allow the next lemma to be applied to the sequence $\{P_i\}$.

LEMMA 3. *Let $\{X_i\}$ be a sequence of subsets of Z. If for each $i \in Z$ there is an s_i such that $s_i \geqq N(X_i, n)/n$ for all n and $\Sigma_{i\in Z} s_i < \infty$, then $D(\bigcup_{i\in Z} X_i) = \lim_{j\to\infty} D(\bigcup_{i\leqq j} X_i)$.*

Proof. Let $Y_1 = X_1$ and $Y_{i+1} = X_{i+1} - \bigcup_{j\leqq i} X_j$. Then the Y_i's are disjoint and Lemma 1 shows that $D(\bigcup_{i\leqq j} X_i) = \Sigma_{i\leqq j} D(Y_i)$. Since $\Sigma_{i\leqq j} D(Y_i)$ is bounded by 1, $\Sigma_{i\in Z} D(Y_i)$ converges. Thus $\lim_{j\to\infty} D(\bigcup_{i\leqq j} X_i) = \Sigma_{i\in Z} D(Y_i) = \Sigma_{i\in Z} \lim_{n\to\infty} N(Y_i, n)/n$. Now $s_i \geqq N(X_i, n)/n \geqq N(Y_i, n)/n$ for all n, so the dominated convergence theorem [**3**, Sec. 1.34] can be applied to obtain

$$\lim_{j\to\infty} D\left(\bigcup_{i\leqq j} X_i\right) = \lim_{n\to\infty} \Sigma_{i\in Z} N(Y_i, n)/n = \lim_{n\to\infty} N\left(\bigcup_{i\in Z} Y_i, n\right)\Big/ n = D\left(\bigcup_{i\in Z} X_i\right).$$

Now since $A_1 = Z - \bigcup_{i\in Z} P_i$, $D(A_1) = 1 - D(\bigcup_{i\in Z} P_i)$. Applying Lemmas 2 and 3 to $D(\bigcup_{i\in Z} P_i)$ yields

$$D\left(\bigcup_{i\in Z} P_i\right) = \lim_{j\to\infty} D\left(\bigcup_{i\leqq j} P_i\right) = \lim_{j\to\infty}\left[1 - \prod_{i\leqq j}(1 - 1/p_i^2)\right] = 1 - \prod_{i\in Z}(1 - 1/p_i^2).$$

As was cleverly shown by Euler [**4**, Sec. 9.11] $D(A_1) = \Pi_{i\in Z}(1 - 1/p_i^2) = 1/(\Sigma_{i\in Z} 1/i^2) = 6/\pi^2$ which is approximately 60.793%. This corresponds very nicely with the data from the first row of TABLE 1. $D(A_1)$ can obviously be approximated by computing $N(A_1, n)/n$ for large n. Also, since $D(A_1)$ in the limit does not depend on the behavior of A_1 in any finite interval, any interval (not just those beginning with 1) can be used to approximate $D(A_1)$. Since the product $\Pi_{i\in Z}(1 - 1/p_i^2)$ converges rapidly, the approximations of $D(A_1)$ made by taking averages over long intervals such as those in TABLE 1 end up being sharp approximations and, therefore, the averages are, to a large part, independent of the intervals.

$D(A_2)$ is only slightly more complicated to calculate than $D(A_1)$. By TABLE 3, the elements in A_2

k	$D(A_k)$	Approximation in %
1	$\frac{6}{\pi^2}$	60.792
2	$\frac{6}{\pi^2}\Sigma\frac{1}{p_i}\left(\frac{1}{p_i+1}\right)$	20.059
3	$\frac{6}{\pi^2}\Sigma\frac{1}{p_i^2}\left(\frac{1}{p_i+1}\right)$	7.412
4	$\frac{3}{\pi^2}\left(\Sigma\frac{1}{p_i}\frac{1}{p_i+1}\right)^2-\frac{3}{\pi^2}\Sigma\frac{1}{p_i^2}\frac{1}{(p_i+1)^2}$	2.207
5	$\frac{6}{\pi^2}\Sigma\frac{1}{p_i^3}\frac{1}{p_i+1}$	3.207
6	$\frac{6}{\pi^2}\Sigma\frac{1}{p_i}\frac{1}{p_i+1}\Sigma\frac{1}{p_j}\frac{1}{p_j+1}-\frac{6}{\pi^2}\Sigma\frac{1}{p_i^3}\frac{1}{(p_i+1)^2}$	1.446
7	$\frac{6}{\pi^2}\Sigma\frac{1}{p_i^4}\frac{1}{p_i+1}$	1.474
8	$\frac{1}{\pi^2}\left(\Sigma\frac{1}{p_i}\frac{1}{p_i+1}\right)^3-\frac{3}{\pi^2}\Sigma\frac{1}{p_i}\frac{1}{p_i+1}\Sigma\frac{1}{p_j^2}\frac{1}{(p_j+1)^2}+\frac{2}{\pi^2}\Sigma\frac{1}{p_i^3}\frac{1}{(p_i+1)^3}$	0.107
9	$\frac{3}{\pi^2}\left(\Sigma\frac{1}{p_i^2}\frac{1}{p_i+1}\right)^2-\frac{3}{\pi^2}\Sigma\frac{1}{p_i^4}\frac{1}{(p_i+1)^2}$	0.216

Precise formulas and numerical approximations for the densities of the sets A_k of integers n for which there are exactly k Abelian groups of order n.

TABLE 4

are those positive integers whose factorization contain exactly one prime squared. Let $Q_j = \{r \in Z: p_i^2 | r \text{ and } p_i^3 \nmid r\}$. Then $A_2 = \bigcup_{j\in Z}(Q_j - \bigcup_{i\neq j} P_i)$. For any set X, let X^* denote $Z - X$. Then $A_2 = \bigcup_{j\in Z}[Q_j \cap (\bigcup_{i\neq j} P_i)^*]$. Clearly, $D(Q_j) = 1/p_j^2 - 1/p_j^3$ and $D[(\bigcup_{i\neq j} P_i)^*] = [\Pi_{i\in Z}(1-1/p_i^2)]/(1-1/p_j^2) = 6/(1-1/p_j^2)\pi^2$. Thus, by arguing as above, one obtains the formula

$$D(A_2) = \Sigma_{j\in Z}\frac{6}{\pi^2}\left(\frac{1}{p_j}\right)\left(\frac{1}{p_j+1}\right).$$

This sum has an approximate value of 20.059%.

The same procedure will yield the formulas for $D(A_3)$, $D(A_5)$, and $D(A_7)$. Formulas for $D(A_4)$, $D(A_6)$, $D(A_8)$, and $D(A_9)$ which are listed in TABLE 4 are slightly more difficult and are left for the reader to prove. All the formulas yield numbers which agree closely with the values in TABLE 1.

References

[1] J. A. Gallian, Computers in group theory, this MAGAZINE, 49 (1976) 69–73.
[2] I. N. Herstein, Topics in Algebra, Xerox, Lexington, 1975.
[3] W. Rudin, Real and Complex Analysis, McGraw-Hill, New York, 1974.
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A Generalization of a Putnam Problem

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The following problem appeared on the 1973 Putnam Examination: *Let $a_1, a_2, \ldots, a_{2n+1}$ be integers such that, if any one of them is removed, those remaining can be divided into two sets of n having equal sums. Prove* $a_1 = a_2 = \cdots = a_{2n+1}$. A proof may be based on special properties of integers. (Show that