Multiplication Operators on the Bloch Space of a Bounded Homogeneous Domain

Joint Mathematics Meeting

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Acknowledgements

- Thanks to John Conway, Sherwin Kouchekian, and William Ross for organizing this special session.

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- R. Allen and F. Colonna, *Multiplication Operators on the Bloch Space of a Bounded Homogeneous Domain*, submitted to the Journal of Computational Methods and Function Theory.

Research Goals

1. Answer the fundamental questions with respect to multiplication operators $M_\psi f = \psi f$ on the Bloch space:
   (a) What symbols induce bounded operators?
   (b) What are estimates on the norm of the bounded operators?
   (c) What symbols induce compact operators?
   (d) What symbols induce isometric operators?
   (e) What are the spectra of the bounded operators?

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   (c) What symbols induce compact operators?
   (d) What symbols induce isometric operators?
   (e) What are the spectra of the bounded operators?
2. We are also building the framework on which operators with symbol can be studied on the Bloch space on a general class of domains in $\mathbb{C}^n$ which include the unit ball $\mathbb{B}_n$ and the polydisk $\mathbb{D}^n$.

Notation

1. D will denote a domain (open and connected) in $\mathbb{C}^n$.

2. $H(D)$ is the set of all holomorphic functions $f : D \to \mathbb{C}$.

3. $H^\infty(D)$ is the space of all bounded holomorphic functions $f : D \to \mathbb{C}$.

4. $\mathrm{Aut}(D)$ is the group of biholomorphic maps from D onto D.

Background

Homogeneous Domains

- A domain $D \subseteq \mathbb{C}^n$ is called **homogeneous** if $\mathrm{Aut}(D)$ acts transitively on D.
- A domain $D \subseteq \mathbb{C}^n$ is called **symmetric** if for every $z_0 \in D$, there exists an involution $\phi \in \mathrm{Aut}(D)$ for which z_0 is an isolated fixed point.

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- Cartan showed that any bounded symmetric domain can be written as a product of irreducible bounded symmetric domains. The irreducible domains fall into one of six classes, four called the **classical domains** and two called the **exceptional domains**.

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- Cartan showed that any bounded symmetric domain can be written as a product of irreducible bounded symmetric domains. The irreducible domains fall into one of six classes, four called the **classical domains** and two called the **exceptional domains**.
- Simple examples of symmetric domains are the unit ball and polydisk.

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- Examples:

1. On the unit ball $\mathbb{B}_n$:

$$H_z(u,\overline{v}) = \frac{n+1}{2}\cdot\frac{(1-||z||^2)\langle u,v\rangle + \langle u,z\rangle\langle z,v\rangle}{(1-||z||^2)^2}.$$

2. On the unit polydisk $\mathbb{D}^n$:

$$H_z(u,\overline{v}) = \sum_{j=1}^{n}\frac{u_j\overline{v_j}}{(1-|z_j|^2)^2}.$$

The **Bloch space** on the unit disk $\mathbb{D}$ is the set $\mathcal{B}(\mathbb{D})$ of functions $f \in H(\mathbb{D})$ such that

$$\beta_f = \sup_{z\in\mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$

$\mathcal{B}(\mathbb{D})$ is a Banach space under the norm

$$||f||_{\mathcal{B}} = |f(0)| + \beta_f.$$

The **little Bloch space** $\mathcal{B}_0(\mathbb{D})$ is the subspace of $\mathcal{B}(\mathbb{D})$ whose elements satisfy

$$\lim_{|z|\to 1} (1 - |z|^2) |f'(z)| = 0.$$

The **Bloch space** on a bounded homogeneous domain D is defined as

$$\mathcal{B}(D) = \left\{ f \in H(D) : \sup_{z \in D} Q_f(z) < \infty \right\},$$

where $Q_f(z) = \sup_{u \in \mathbb{C}^n \setminus \{0\}} \frac{\left| \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(z) u_j \right|}{H_z(u, \overline{u})^{1/2}}$,

with norm

$$||f||_{\mathcal{B}} = |f(0)| + \sup_{z \in D} Q_f(z).$$

The Bloch Space in Higher Dimensions

The **Bloch space** on a bounded homogeneous domain D is defined as

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with norm

$$||f||_{\mathcal{B}} = |f(0)| + \sup_{z \in D} Q_f(z).$$

For a bounded symmetric domain D, the closure of the polynomials in $\mathcal{B}(D)$ is a subspace called the **little Bloch space**, denoted by $\mathcal{B}_0(D)$.

Previously Known Results

Theorem (Arazy, 1982). The function $\psi \in H(\mathbb{D})$ induces a bounded multiplication operator M_ψ on the Bloch space if and only if $\psi \in H^\infty(\mathbb{D})$ and

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)\, |\psi'(z)| \log \frac{1 + |z|}{1 - |z|} < \infty.$$

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Theorem (Brown & Shields, 1991). The function $\psi \in H(\mathbb{D})$ induces a bounded multiplication operator M_ψ on the Bloch space if and only if $\psi \in H^\infty(\mathbb{D})$ and

$$|\psi'(z)| = O\left(\frac{1}{(1-|z|)\log\frac{1}{1-|z|}}\right).$$

Compactness on $\mathcal{B}(\mathbb{D})$

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Theorem (Ohno & Zhao, 2001). Let $\psi \in H(\mathbb{D})$. Then the following are equivalent:

1. M_ψ is compact on $\mathcal{B}$;
2. M_ψ is compact on $\mathcal{B}_0$;
3. $\psi \equiv 0$.

Theorem (A. & Colonna, 2008). Let $\psi \in H(\mathbb{D})$ for which M_ψ is bounded on $\mathcal{B}(\mathbb{D})$. Then

1. $\max\{||\psi||_{\mathcal{B}}, ||\psi||_\infty\} \leq ||M_\psi|| \leq \max\{||\psi||_{\mathcal{B}}, ||\psi||_\infty + \sigma_\psi\},$
   where
$$\sigma_\psi = \sup_{z\in\mathbb{D}} \frac{1}{2}(1-|z|^2)\left|\psi'(z)\right| \log\frac{1+|z|}{1-|z|}.$$

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2. $\sigma(M_\psi) = \overline{\psi(\mathbb{D})}$.
3. M_ψ is an isometry on $\mathcal{B}(\mathbb{D})$ if and only if ψ is a constant function of modulus 1.

The Bloch space on the ball can also be defined as the Banach space of functions $g \in H(\mathbb{B}_n)$ for which $\sup_{z \in \mathbb{B}_n} (1 - \|z\|^2) \|\nabla g(z)\| < \infty$ under the norm

$$\|f\|_* = |f(0)| + \sup_{z \in \mathbb{B}_n} (1 - \|z\|^2) \|\nabla f(z)\|.$$

The Bloch space on the ball can also be defined as the Banach space of functions $g \in H(\mathbb{B}_n)$ for which $\sup_{z \in \mathbb{B}_n} (1 - ||z||^2) ||\nabla g(z)|| < \infty$ under the norm

$$||f||_* = |f(0)| + \sup_{z \in \mathbb{B}_n} (1 - ||z||^2) ||\nabla f(z)|| .$$

Theorem (Zhu, 2004). For a function $\psi \in H(\mathbb{B}_n)$, the following conditions are equivalent:

1. M_ψ is bounded on $\mathcal{B}(\mathbb{B}_n)$;
2. M_ψ is bounded on $\mathcal{B}_0(\mathbb{B}_n)$;
3. $\psi \in H^\infty(\mathbb{B}_n)$ and
$$(1 - ||z||^2) ||\nabla f(z)|| \log \frac{1}{1 - ||z||^2} < \infty.$$

Current Research

Let D be a bounded homogeneous domain. For $z \in D$, define

$$\omega(z) = \sup\{|f(z)| : f \in \mathcal{B}(D), f(0) = 0, \text{ and } ||f||_{\mathcal{B}} \leq 1\}$$

$$\sigma_\psi = \sup_{z \in D} \omega(z) Q_\psi(z).$$

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$$\sigma_\psi = \sup_{z \in D} \omega(z) Q_\psi(z).$$

Let D be a bounded symmetric domain. For $z \in D$, define

$$\omega_0(z) = \sup\{|f(z)| : f \in \mathcal{B}_0(D), f(0) = 0, \text{ and } ||f||_{\mathcal{B}} \leq 1\}$$
$$\sigma_{0,\psi} = \sup_{z \in D} \omega_0(z) Q_\psi(z).$$

Facts.

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2. $\omega(z) = \sup_{f \in \mathcal{E}} |f(z)|$ where $\mathcal{E}$ is the set of extreme points of $\{f \in \mathcal{B}(D) : f(0) = 0$ and $||f||_{\mathcal{B}} \leq 1\}$.

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3. If $D = \mathbb{B}_n$, then $\omega_0(z) = \omega(z) = \rho(0,z) = \frac{1}{2}\log\frac{1+||z||}{1-||z||}$ for all $z \in \mathbb{B}_n$.

Theorem (A. & Colonna). Let D be a bounded homogeneous domain and $\psi \in H(D)$.

(a) Then M_ψ is bounded on $\mathcal{B}(D)$ if and only if $\psi \in H^\infty(D)$ and $\sigma_\psi < \infty$.

(b) If D is a bounded symmetric domain, then M_ψ is a bounded operator from $\mathcal{B}_0(D)$ into $\mathcal{B}(D)$ if and only if $\psi \in H^\infty(D)$ and $\sigma_{0,\psi} < \infty$.

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Special Case. If $\psi \in H(\mathbb{B}_n)$, then the following are equivalent:

1. M_ψ is bounded on $\mathcal{B}(\mathbb{B}_n)$;
2. M_ψ is bounded on $\mathcal{B}_0(\mathbb{B}_n)$;
3. $\psi \in H^\infty(\mathbb{B}_n)$ and $\sup_{z \in \mathbb{B}_m} \log \frac{1+||z||}{1-||z||} Q_\psi(z) < \infty$.

Theorem (A. & Colonna). Let D be a bounded homogeneous domain in $\mathbb{C}^n$ and $\psi \in H(D)$.

1. If M_ψ is bounded on $\mathcal{B}(D)$, then
$$\max\{||\psi||_{\mathcal{B}}, ||\psi||_\infty\} \leq ||M_\psi|| \leq \max\{||\psi||_{\mathcal{B}}, ||\psi||_\infty + \sigma_\psi\}.$$
2. If D is a bounded symmetric domain and M_ψ is bounded from $\mathcal{B}_0(D)$ into $\mathcal{B}(D)$, then
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Special Case. If $\psi \in H(\mathbb{B}_m)$ induces a bounded multiplication operator on $\mathcal{B}(\mathbb{B}_n)$, then

$$||M_\psi|| \geq \max\{||\psi||_{\mathcal{B}}, ||\psi||_\infty\}$$

$$||M_\psi|| \leq \max\left\{||\psi||_{\mathcal{B}}, ||\psi||_\infty + \tfrac{1}{2} \sup_{z\in\mathbb{B}_n} \log \frac{1+||z||}{1-||z||} Q_\psi(z)\right\}.$$

Spectrum & Compact Operators

Theorem (A. & Colonna). Let D be a bounded homogeneous (symmetric) domain in $\mathbb{C}^n$ and assume $\psi \in H(D)$ induces a bounded multiplication operator on $\mathcal{B}(D)$ ($\mathcal{B}_0(D)$). Then $\sigma(M_\psi) = \overline{\psi(D)}$.

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Theorem (A. & Colonna). Let $D = D_1 \times \cdots \times D_k$ where each D_j is an irreducible Cartan domain other than $\mathbb{D}$ and $\psi \in H(D)$. Then M_ψ is an isometry on $\mathcal{B}(D)$ if and only if ψ is a constant function of modulus one.

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As stated previously, the isometric multiplication operators on $\mathcal{B}(\mathbb{D})$ are known to be induced by constant functions of modulus one.

Future Research

Open Questions

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Conjecture. If D is a bounded homogeneous domain in $\mathbb{C}^n$ and $\psi \in H(D)$ is the symbol of a bounded multiplication operator on $\mathcal{B}(D)$, then M_ψ is an isometry on $\mathcal{B}(D)$ if and only if ψ is a constant function of modulus one.

Weighted Composition Operators

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Stay Tuned

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Flavia Colonna, 4:30 pm today!

Weighted Composition Operators on the Bloch space in $\mathbb{C}^n$