Weighted Composition Operators from H^∞ to the Bloch Space on a Bounded Homogeneous Domain

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   - A domain $D \subseteq \mathbb{C}^n$ is called **homogeneous** if for every $z, w \in D$, there exists and automorphism ϕ of D such that $\phi(z) = w$.
   - A domain $D \subseteq \mathbb{C}^n$ is called **symmetric** if for every $z_0 \in D$, there exists an involutive automorphism ϕ of D for which z_0 is an isolated fixed point.

The Bloch Space

- The **Bloch space** on a bounded homogeneous domain D, denoted $\mathcal{B}(D)$ is the space of holomorphic function $f : D \to \mathbb{C}$ such that

$$\beta_f = \sup_{z \in D} Q_f(z) < \infty,$$

where

$$Q_f(z) = \sup_{u \in \mathbb{C}^n \setminus \{0\}} \frac{\left| \sum_{j=1}^n \frac{\partial f}{\partial z_j}(z) u_j \right|}{H_z(u, \overline{u})^{1/2}},$$

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- $\mathcal{B}(D)$ is a Banach space under the norm $||f||_{\mathcal{B}} = |f(0)| + \beta_f$.

- $H^\infty(D)$ is the space of bounded holomorphic functions on D with norm $||f||_\infty = \sup_{z \in D} |f(z)|$.

- $H^\infty(D)$ is a proper subspace of $\mathcal{B}(D)$ since $||f||_{\mathcal{B}} \leq 2\,||f||_\infty$.

Previous Research

- Ohno studied $W_{\psi,\varphi} : H^\infty(\mathbb{D}) \to \mathcal{B}(\mathbb{D})$.
  1. characterization of the bounded $W_{\psi,\varphi}$.
  2. characterization of the compact $W_{\psi,\varphi}$.
- Li and Stević studied $W_{\psi,\varphi} : H^\infty(\mathbb{D}^n) \to \mathcal{B}(\mathbb{D}^n)$.
  1. characterization of the bounded $W_{\psi,\varphi}$.
  2. characterization of the compact $W_{\psi,\varphi}$.
- Li and Stević also studied $W_{\psi,\varphi} : H^\infty(\mathbb{B}_n) \to \mathcal{B}(\mathbb{B}_n)$.
  1. necessary conditions and sufficient conditions for bounded $W_{\psi,\varphi}$.
  2. necessary conditions and sufficient conditions for compact $W_{\psi,\varphi}$.

Boundedness

Let D be a bounded homogeneous domain, $\psi : D \to \mathbb{C}$ holomorphic, and $\varphi = (\varphi_1, \ldots, \varphi_n)$ a holomorphic self-map of D. Define for $z \in D$,

$$\theta_\varphi(z) = \sup\left\{Q_{f\circ\varphi}(z) : f \in H^\infty(D), ||f||_\infty \leq 1\right\}$$
$$\theta_{\psi,\varphi} = \sup_{z\in D} |\psi(z)|\,\theta_\varphi(z).$$

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Theorem

The weighted composition operator $W_{\psi,\varphi} : H^\infty(D) \to \mathcal{B}(D)$ is bounded if and only if $\psi \in \mathcal{B}(D)$ and $\theta_{\psi,\varphi}$ is finite.

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This provides a complete characterization of the bounded weighted composition operators from $H^\infty(\mathbb{B}_n)$ to $\mathcal{B}(\mathbb{B}_n)$.

Operator Norm Estimates

Theorem

If $W_{\psi,\varphi} : H^\infty(D) \to \mathcal{B}(D)$ *is bounded, then*

$$||\psi||_{\mathcal{B}} \leq ||W_{\psi,\varphi}|| \leq ||\psi||_{\mathcal{B}} + \theta_{\psi,\varphi}.$$

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If $D = \mathbb{D}$, then we have

$$\theta_{\psi,\varphi} = \sup_{z \in \mathbb{D}} |\psi(z)| \frac{(1-|z|^2)\,|\varphi'(z)|}{1-|\varphi(z)|^2}.$$

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Corollary

If $W_{\psi,\varphi} : H^\infty(\mathbb{D}) \to \mathcal{B}(\mathbb{D})$ *is bounded, then*

$$\max\{||\psi||_{\mathcal{B}}, \theta_{\psi,\varphi}\} \leq ||W_{\psi,\varphi}|| \leq ||\psi||_{\mathcal{B}} + \theta_{\psi,\varphi}.$$

Compactness

Theorem

Let $\psi \in \mathcal{B}(D)$*. Then* $W_{\psi,\varphi} : H^\infty(D) \to \mathcal{B}(D)$ *is compact if*

$$\lim_{\varphi(z)\to\partial D} Q_\psi(z) = 0 \text{ and } \lim_{\varphi(z)\to\partial D} |\psi(z)|\, \theta_\varphi(z) = 0.$$

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Theorem

Let ψ be holomorphic on $\mathbb{B}_n$ and φ be a holomorphic self-map of $\mathbb{B}_n$. Then the bounded operator $W_{\psi,\varphi} : H^\infty(\mathbb{B}_n) \to \mathcal{B}(\mathbb{B}_n)$ is compact if and only if

$$\lim_{\varphi(z)\to\partial\mathbb{B}_n} Q_\psi(z) = 0 \text{ and } \lim_{\varphi(z)\to\partial\mathbb{B}_n} |\psi(z)|\, \theta_\varphi(z) = 0.$$

Theorem

Let φ be a holomorphic self-map of $\mathbb{D}^n$ and $\psi \in \mathcal{B}(\mathbb{D}^n)$. Then $W_{\psi,\varphi} : H^\infty(\mathbb{D}^n) \to \mathcal{B}(\mathbb{D}^n)$ is compact if and only if

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1. $M_\psi : H^\infty(D) \to \mathcal{B}(D)$ is bounded if and only if $\psi \in H^\infty(D)$.
2. $\max\{||\psi||_{\mathcal{B}}, c_D\,||\psi||_\infty\} \leq ||M_\psi|| \leq ||\psi||_{\mathcal{B}} + c_D\,||\psi||_\infty$, where c_D is the Bloch constant of D, defined by

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$$c_D = \sup\{\beta_f : f : D \to \mathbb{C} \text{ holomorphic, and } f(D) \subseteq \mathbb{D}\}.$$

3. If D has the unit disk $\mathbb{D}$ as a factor, then
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5. If $D = \mathbb{D}$, then M_ψ is not an isometry from $H^\infty(\mathbb{D})$ to $\mathcal{B}(\mathbb{D})$.

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3. If $D = \mathbb{D}$, then C_φ is not an isometry from $H^\infty(\mathbb{D})$ to $\mathcal{B}(\mathbb{D})$.

Further Directions

- Complete the characterization of compactness of $W_{\psi,\varphi} : H^\infty(D) \to \mathcal{B}(D)$ for a bounded homogeneous domain D.
- Find an isometric $W_{\psi,\varphi} : H^\infty(\mathbb{D}) \to \mathcal{B}(\mathbb{D})$, or prove none exist.
- Determine if isometric $W_{\psi,\varphi} : H^\infty(D) \to \mathcal{B}(D)$ exist for other bounded domains D.