The Bloch Space

Function- & Operator-Theoretic Perspectives

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Graduate Seminar
September 19, 2006

Outline

Preliminaries

Notation

- Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk in $\mathbb{C}$.
- Let $H(\mathbb{D})$ denote the set of analytic function on $\mathbb{D}$.
- The sup-norm $||\cdot||_\infty$ on $H(\mathbb{D})$ is defined as $||f||_\infty = \sup_{z\in\mathbb{D}} |f(z)|$.
- Let $H^\infty(\mathbb{D}) = \{f \in H(\mathbb{D}) : ||f||_\infty < \infty\}$ denote the set of bounded analytic functions on $\mathbb{D}$.

Schwarz-Pick Lemma

Let $f : \mathbb{D} \to \overline{\mathbb{D}}$ be analytic. Then for $z \in \mathbb{D}$,

$$(1 - |z|^2)\left|f'(z)\right| \leq 1 - |f(z)|^2.$$

If $f(z)$ is a conformal automorphism of $\mathbb{D}$, then equality holds; otherwise the inequality is strict for all $z \in \mathbb{D}$.

Bloch Functions & the Bloch Space

Definition

A function $f \in H(\mathbb{D})$ is said to be Bloch provided

$$\beta_f := \sup_{z \in \mathbb{D}} (1 - |z|^2) \left|f'(z)\right| < \infty.$$

The Bloch space is defined as $\mathcal{B} = \{f \in H(\mathbb{D}) : \beta_f < \infty\}$.

Immediate Consequences

- If f is Bloch, then so is αf for $\alpha \in \mathbb{C}$ and

$$\beta_{\alpha f} = \sup_{z \in \mathbb{D}} (1 - |z|^2) \left|(\alpha f)'(z)\right| = \sup_{z \in \mathbb{D}} (1 - |z|^2) |\alpha| \left|f'(z)\right| = |\alpha| \beta_f.$$

- If f and g are Bloch, then so is $f + g$ and

$$\begin{aligned} \beta_{f+g} &= \sup_{z \in \mathbb{D}} (1 - |z|^2) \left|f'(z) + g'(z)\right| \leq \sup_{z \in \mathbb{D}} (1 - |z|^2) \left(\left|f'(z)\right| + \left|g'(z)\right|\right) \\ &\leq \sup_{z \in \mathbb{D}} (1 - |z|^2) \left|f'(z)\right| + \sup_{z \in \mathbb{D}} (1 - |z|^2) \left|g'(z)\right| = \beta_f + \beta_g. \end{aligned}$$

Examples

- Let $f(z) = \alpha$ for $\alpha \in \mathbb{C}$.

$$\beta_f = \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| f'(z) \right| = 0.$$

Hence $f \mapsto \beta_f$ is *not a norm*, but a seminorm, on the Bloch space.

- Let $f(z) = z$.

$$\beta_f = \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| f'(z) \right| = \sup_{z \in \mathbb{D}} (1 - |z|^2) = 1.$$

- Let $n \geq 2$ and $f(z) = z^n$.

$$\begin{aligned} \beta_f &= \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| f'(z) \right| = \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| n \right| \left| z \right|^{n-1} \\ &= |n| \sup_{x \in [0,1)} (x^{n-1} - x^{n+1}) \leq |n| \,. \end{aligned}$$

- Any analytic function $f : \mathbb{D} \to \overline{\mathbb{D}}$ is Bloch.

  By the Schwarz-Pick lemma

$$\beta_f = \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| f'(z) \right| \leq \sup_{z \in \mathbb{D}} (1 - |f(z)|^2) \leq 1.$$

Examples

1. For $a \in \mathbb{D}$ and $|\lambda| = 1$, the Möbius Transformations of $\mathbb{D}$ onto $\mathbb{D}$ are

$$\varphi_a(z) = \lambda \frac{a - z}{1 - \overline{a} z}.$$

2. Let $\{a_n\}$ be a sequence in $\mathbb{D}$ satisfying the Blaschke condition of $\sum_n (1 - |a_n|) < \infty$. We define the Blaschke product as

$$B(z) = z^{m_0} \prod_k \left(\frac{\overline{a_k}}{|a_k|} \frac{a_k - z}{1 - \overline{a_k} z} \right)^{m_k},$$

   where $m_i \in \mathbb{Z}_+$.

- Let $f(z) = \log\left(\frac{1+z}{1-z}\right)$.

$$\begin{aligned}\beta_f &= \sup_{z\in\mathbb{D}}(1-|z|^2)\left|f'(z)\right| = \sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{2}{(1+z)(1-z)}\right| \\ &= \sup_{z\in\mathbb{D}}(1-|z|^2)\left(\frac{2}{|1-z^2|}\right) \leq \sup_{z\in\mathbb{D}}(1-|z|^2)\left(\frac{2}{1-|z|^2}\right) \\ &\leq 2.\end{aligned}$$

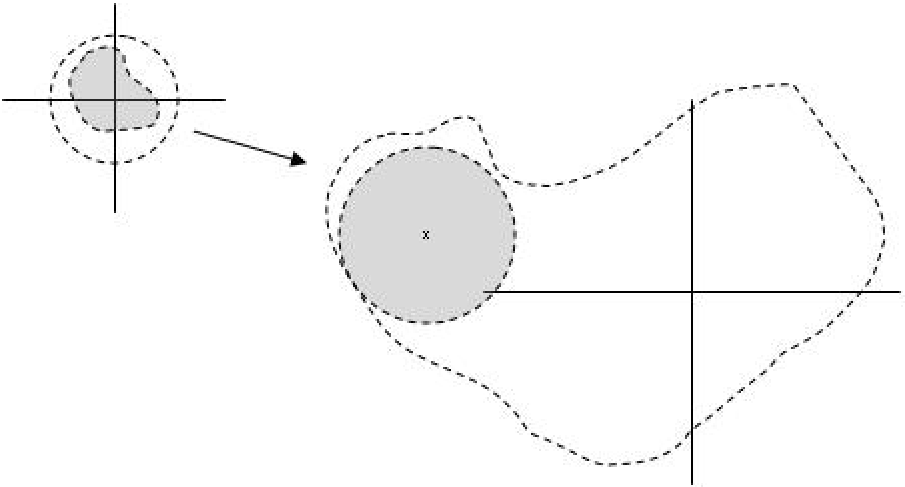
Note

f is unbounded in the sup-norm. Consider the sequence $z_n = \frac{1}{n} - 1$. Clearly $|f(z_n)| \to \infty$ as $n \to \infty$.

Geometric Characterization of Bloch functions

Definition

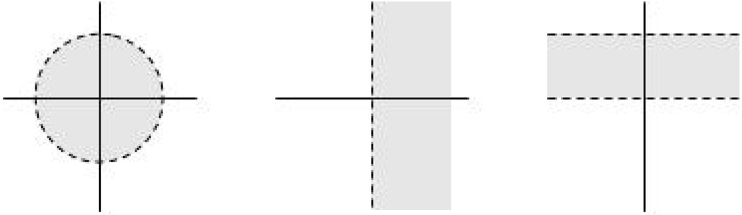
A schlicht disk in $f(\mathbb{D})$ is an open disk $\Delta \subset f(\mathbb{D})$ such that there exists a domain $\Omega \subseteq \mathbb{D}$ with f mapping Ω bijectively onto Δ.



- For any $z \in \mathbb{D}$, let $d_f(z)$ be the radius of the largest schlicht disk in $f(\mathbb{D})$ centered at $f(z)$. f is Bloch if and only if

$$\sup_{z\in\mathbb{D}} d_f(z) < \infty.$$

- We see geometrically why $f(z) = \log\left(\frac{1+z}{1-z}\right)$ is Bloch.



- For $f \in \mathcal{B}$ and $0 < r < 1$, we see that $f_r(z) = f(rz) \in \mathcal{B}$.

Proposition

For $f \in \mathcal{B}$ and φ analytic from $\mathbb{D}$ into $\mathbb{D}$, then

$$\beta_{f\circ\varphi} \leq \beta_f.$$

Moreover, if φ is a conformal automorphism of $\mathbb{D}$, then equality holds; $\mathcal{B}$ is Möbius Invariant.

Proof. Since φ is analytic from $\mathbb{D}$ into $\mathbb{D}$, the Schwarz-Pick lemma asserts $(1-|z|^2)\,|\varphi'(z)| \leq 1-|\varphi(z)|^2$ for all $z \in \mathbb{D}$. So, we have

$$\begin{aligned}
\beta_{f\circ\varphi} &= \sup_{z\in\mathbb{D}}(1-|z|^2)\left|(f\circ\varphi)'(z)\right| = \sup_{z\in\mathbb{D}}(1-|z|^2)\left|\varphi'(z)\right|\left|f'(\varphi(z))\right| \\
&\leq \sup_{z\in\mathbb{D}}(1-|\varphi(z)|^2)\left|f'(\varphi(z))\right| = \sup_{w\in\varphi(\mathbb{D})}(1-|w|^2)\left|f'(w)\right| \\
&\leq \sup_{w\in\mathbb{D}}(1-|w|^2)\left|f'(w)\right| = \beta_f.
\end{aligned}$$

If φ is conformal automorphism of $\mathbb{D}$, then $(1-|z|^2)\,|\varphi'(z)| = 1-|\varphi(z)|^2$ for all $z \in \mathbb{D}$ and $\varphi(\mathbb{D}) = \mathbb{D}$. So equality holds throughout.

Proposition

For all $f \in H(\mathbb{D})$, $\beta_f \leq ||f||_\infty$, in the extended sense.

Proof. If $||f||_\infty = 0$, then the inequality holds since $f \equiv 0$ and $\beta_f = 0$.

If $||f||_\infty = \infty$, then the inequality holds as well.

Suppose $0 < ||f||_\infty < \infty$. Note that $g(z) = \frac{f(z)}{||f||_\infty}$ has $||g||_\infty = 1$, and thus maps $\mathbb{D}$ into $\overline{\mathbb{D}}$. The Schwarz-Pick lemma asserts

$$\frac{1}{||f||_\infty}\beta_f = \beta_g = \sup_{z\in\mathbb{D}}(1 - |z|^2)\left|g'(z)\right| \leq \sup_{z\in\mathbb{D}}(1 - |g(z)|^2) \leq 1.$$

So, we have $\beta_f \leq ||f||_\infty$.

Corollary

$H^\infty(\mathbb{D})$ is properly contained in $\mathcal{B}$.

Linear Structure

Definition

A normed linear space X is a vector space (over $\mathbb{C}$) with norm $||\cdot|| : X \to [0, \infty)$ such that

1. $||x|| = 0$ *iff* $x = 0$,
2. $||x + y|| \leq ||x|| + ||y||$,
3. $||\alpha x|| = |\alpha| \, ||x||$.

The Bloch space is a normed linear space with respect to the norm

$$||f|| = \beta_f + |f(0)| \,.$$

Proposition

For $f \in \mathcal{B}$ and $z \in \mathbb{D}$

$$|f(z) - f(0)| \leq \frac{1}{2}\beta_f \log\left(\frac{1+|z|}{1-|z|}\right).$$

Proof.

$$\begin{aligned}
|f(z) - f(0)| &= \left|\int_0^1 z \cdot f'(zt)\, dt\right| \leq \int_0^1 |z|\left|f'(zt)\right|\, dt \\
&= |z|\int_0^1 \frac{(1-|zt|^2)\left|f'(zt)\right|}{1-|zt|^2}\, dt \leq |z|\int_0^1 \frac{\beta_f}{1-|z|^2 t^2}\, dt \\
&= |z|\,\beta_f\left[\int_0^1 \frac{1/2}{1+|z|\,t}\, dt + \int_0^1 \frac{1/2}{1-|z|\,t}\, dt\right] \\
&= \frac{1}{2}\beta_f \log\left(\frac{1+|z|\,t}{1-|z|\,t}\right)\Bigg|_0^1 = \frac{1}{2}\beta_f \log\left(\frac{1+|z|}{1-|z|}\right).
\end{aligned}$$

Proposition

For $f \in \mathcal{B}$ and $z \in \mathbb{D}$ with $|z| \geq \frac{1}{2}$, $|f(z)| \leq ||f|| \log \dfrac{1+|z|}{1-|z|}$.

Proof. For $z \in \mathbb{D}$ with $|z| \geq \frac{1}{2}$, we have

$$|z| \geq \frac{1}{2} \geq \frac{e-1}{e+1} \Longrightarrow \frac{1+|z|}{1-|z|} \geq e \Longrightarrow \log\left(\frac{1+|z|}{1-|z|}\right) \geq 1$$

$$\Longrightarrow |f(0)| \leq |f(0)| \log\left(\frac{1+|z|}{1-|z|}\right)$$

$$\begin{aligned} |f(z)| &\leq |f(z) - f(0)| + |f(0)| \\ &\leq \beta_f \log\left(\frac{1+|z|}{1-|z|}\right) + |f(0)| \log\left(\frac{1+|z|}{1-|z|}\right) \\ &= ||f|| \log\left(\frac{1+|z|}{1-|z|}\right). \end{aligned}$$

Corollary

$\Lambda_w f = f(w)$ is a bounded linear functional on the Bloch space.

Proposition

The Bloch space is a Banach space.

Sketch Proof. It suffices to show that any Cauchy sequence $\{f_n\} \subseteq \mathcal{B}$ converges with respect to $||\cdot||$.

1. The sequence of dilations $\{f_{n,r}(z) = f_n(rz)\}$ is Cauchy with respect to $||\cdot||_\infty$. So $f_{n,r} \to f_r$ uniformly on $\overline{\mathbb{D}}$.
2. Let $f(z) = f_r\left(\frac{z}{r}\right)$ for $|z| \leq r < 1$. $f_{n,r}\left(\frac{z}{r}\right) = f_n(z) \to f(z)$ locally uniformly on $\mathbb{D}$. So f is analytic.
3. Lastly, we need to show $||f_n - f|| \to 0$. It suffices to show that $\beta_{f_n - f} \to 0$ since $||f_n - f|| = |f_n(0) - f(0)| + \beta_{f_n - f}$.
    - $\{\beta_{f_n}\}$ is Cauchy since $\{||f_n||\}$ and $\{|f(0)|\}$ are Cauchy.
    - $f_n \to f$ uniformly on compact subsets of $\mathbb{D} \implies f_n' \to f'$ locally uniformly on $\mathbb{D}$.

Relations to Other Analytic Function Spaces

Definition

Let $0 < p < \infty$. *The Bergman spaces are a 1-parameter family of Hilbert spaces*

$$A^p = L^p_a(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \left(\int_{\mathbb{D}} |f(z)|^p \ dA \right)^{1/p} < \infty \right\}.$$

Recall

The dual space of a Banach space X is the space X^* of bounded linear functionals on X.

$(A^1)^*$ can be identified with the Bloch space. Each functional $\varphi \in (A^1)^*$ has the unique representation

$$\varphi(f) = \varphi_g(f) = \lim_{t \to 1} \int_{t\mathbb{D}} f\overline{g} \ d\sigma$$

for $f \in A^1$ and $g \in \mathcal{B}$.

Definition

Let $0 < p < \infty$. The Besov spaces are a 1-parameter family of Hilbert spaces

$$B_p(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \left(\int_{\mathbb{D}} (1 - |z|)^{p-2} \left| f'(z) \right|^p \, dA \right)^{1/p} < \infty \right\}.$$

The Dirichlet space is

$$\mathcal{D} = B_2(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \left(\int_{\mathbb{D}} \left| f'(z) \right|^2 \, dA \right)^{1/2} < \infty \right\}$$

- For $p \geq 1$, $B_p \subset \mathcal{B}$.
- The Dirichlet integral of f represents the area of $f(\mathbb{D})$. So for an analytic function f whose image $f(\mathbb{D})$ has finite area, f is Bloch.

Composition Operators on the Bloch Space

We define the composition operator on $\mathcal{B}$ by $C_\varphi(f) = f \circ \varphi$, with operator norm

$$||C_\varphi|| = \sup_{\substack{f \in \mathcal{B} \\ ||f||=1}} ||C_\varphi(f)|| \,.$$

Corollary

If φ is an analytic self-map of $\mathbb{D}$, C_φ is a bounded linear operator from $\mathcal{B}$ to $\mathcal{B}$.

This all follows from the fact that $\beta_{f\circ\varphi} \leq \beta_f$ for $f \in \mathcal{B}$ and φ an analytic self-map of $\mathbb{D}$.

Question:

For what symbol φ is C_φ an isometry, i.e. $||C_\varphi(f)|| = ||f||$ for all $f \in \mathcal{B}$?

Answer: [Colonna, 2005]

The analytic functions φ from $\mathbb{D}$ into itself which induce an isometric composition operator on the Bloch space are precisely:

1. $\varphi = \lambda z$ with $|\lambda| = 1$; the rotations of the identity
2. $\varphi = gB$ where g is a nonvanishing analytic function of $\mathbb{D}$ into $\overline{\mathbb{D}}$, and B is a Blaschke product whose zeros form an infinite sequence $\{z_n\}$ containing 0 and an infinite subsequence $\{z_{n_j}\}$ such that $\left|g(z_{n_j})\right| \to 1$ and
$$\lim_{j\to\infty} \prod_{k\neq n_j} \left| \frac{z_{n_j} - z_k}{1 - \overline{z_{n_j}} z_k} \right| = 1.$$

Further Considerations

1. Subspaces of the Bloch Space
   - Little Bloch Space $\mathcal{B}_0 = \left\{ f \in \mathcal{B} : \lim_{|z| \to 1} (1 - |z|^2) |f'(z)| = 0 \right\}$.
   - $\mathcal{B}_1 = \{f \in \mathcal{B} : f(0) = 0$ and $||f|| \leq 1.\}$
2. Weighted Bloch Space
   $$\mathcal{B}_{\log} = \left\{ f \in H(\mathbb{D}) : \sup_{z \in \mathbb{D}} (1 - |z|^2) \left(\log \frac{2}{1 - |z|^2} \right) |f'(z)| < \infty \right\}$$
3. Bloch Space in Higher Dimensions
   - Bloch Space on the Polydisk $\mathbb{D}^n$.
   - Bloch Space on the Unit Ball $\mathbb{B}^n$.

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