

Analysis on Discrete Structures

The Beginnings of Discrete Operator Theory?

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Interpolation Problem

- Can we build a bounded analytic function on the entire disk from knowing the values on a discrete set of points in the disk?
- Is the function on the disk unique?

Adgenda

1 Trees

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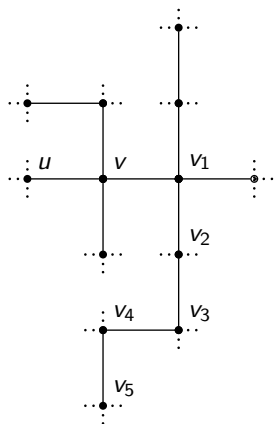
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- 2 Topological Structure of Trees
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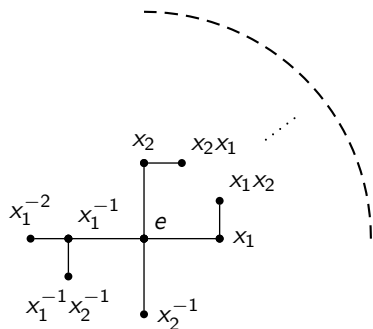
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- 8 Research Directions



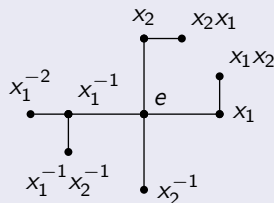
- A *Tree* is an infinite, acyclic, connected graph. A tree is said to be *homogeneous* if every vertex has the same number of edges, denoted by d . The number d is called the *degree* of the tree.
- Vertices u and v are *associate*, $u \sim v$, if there is an edge connecting them.
- A *path* is a sequence of vertices $[v_1, v_2, v_3, v_4, v_5, \dots]$ where $v_i \sim v_{i+1}$ and $v_i \neq v_{i+2}$.
- Define $d_T(u, v)$ to be the length of the path connecting u and v . Then d_T is a *metric* on T .

Topological Structure of Trees



- Let \cong be the equivalence relation on the set of infinite paths generated by $[v_0, v_1, \dots] \cong [v_1, v_2, \dots]$.
- An **end** of T is an equivalence class under \cong . The set of ends of T is denoted by Ω .
- We can define a topology on Ω for which it is Hausdorff, compact and totally disconnected.
- If we define $T^* = T \cup \Omega$, then T^* is sequentially compact, which implies T^* is compact.

Trees as Cayley Graphs



- Let $X = \{x_1, x_2\}$ and F_X be the free group with generators X .
- F_X is made up of “words” comprised of the letters $\{x_1, x_2, x_1^{-1}, x_2^{-1}\}$. We call e the *empty word*.
- Since F_X is free, then the *Cayley Graph* is a rooted tree of degree 4 with root e .
- We say that the tree T is generated by $\{x_1, x_2\}$.

NB! The trees discussed in this talk are of even degree. Trees of odd degree can be constructed with some technical changes, but all results will be the same.

Translation to the Root

Let $h \in F_X$. We can define the map $\lambda_h : T \rightarrow T$ as follows:

$$\lambda_h(v) = hv.$$

This map makes sense when we think of the vertices of T as being elements in F_X . The map λ_h is a **left group action** of F_X on T . The effect of λ_h is to translate the vertex h to the root of T

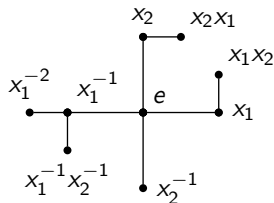
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Tree T

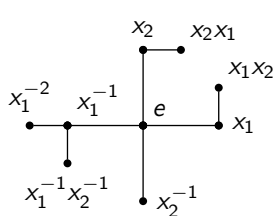
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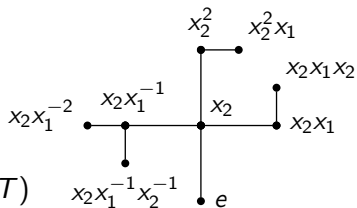
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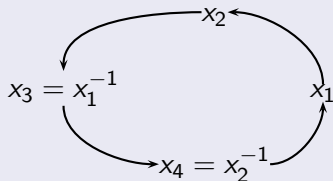
Tree T



Tree $\lambda_{x_2}(T)$

Rotation about the Root

Consider the tree T generated by $\{x_1, x_2\}$. The set of generators and inverses is thus $\{x_1, x_2, x_1^{-1}, x_2^{-1}\}$. If we think of this as a cycle

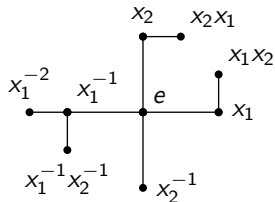


then we can define the map $\psi(x_j) = x_{j+1}$. We can extend the map ψ to a left action on T by defining $\psi(x_i x_j) = x_{i+1} x_{j+1}$.

The map ψ is called the **rotation action about the root**.

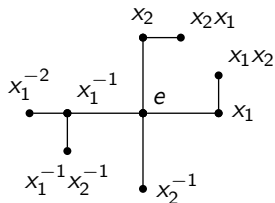
Example of Rotation about the Root

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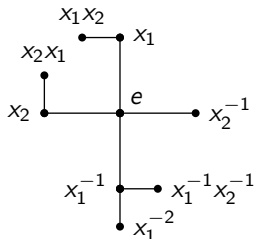


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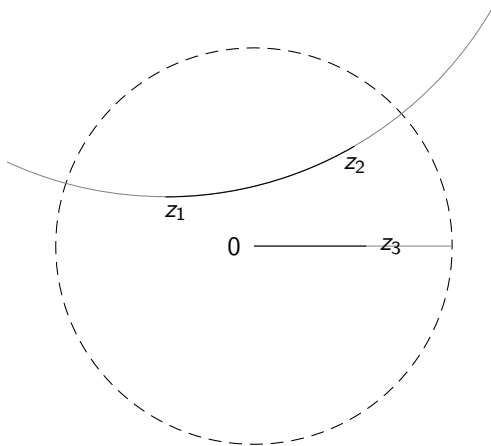


Then applying ψ to T , we have the following rotated tree



Hyperbolic Geometry of the Disk

In order to “fit” an infinite tree inside the unit disk, we must use a geometry other than Euclidean. For this job, the **hyperbolic geometry** on the disk is ideal.



- The **hyperbolic metric** on the unit disk is given by

$$\rho(z_1, z_2) = 2 \tanh^{-1} \left| \frac{z_1 - z_2}{1 - z_1 \overline{z_2}} \right|$$

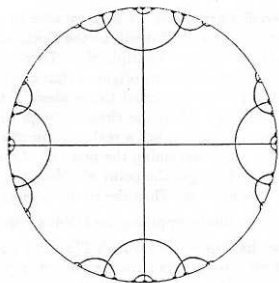
$$\rho(0, z_2) = 2 \tanh^{-1} |z_2|.$$

- The shortest distance between two points z_1 and z_2 is the arc of the circle passing through z_1 and z_2 that is orthogonal to the unit circle.

Embeddings of Trees into the Disk

Let $\Phi : T \rightarrow \mathbb{D}$ be a map which sends vertices of T to points in \mathbb{D} . This map is an embedding of T into \mathbb{D} if it satisfies the following rules:

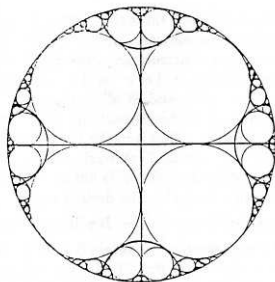
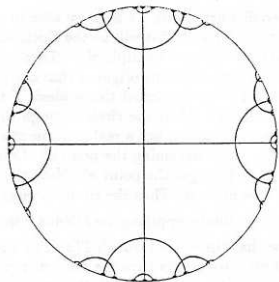
- The root of T is mapped to 0.
- The edges of T are mapped to geodesics in \mathbb{D} under the hyperbolic metric.
- Each edge is to have equal (hyperbolic) length, denoted by r .
- No infinite paths intersect excepts at the origin in \mathbb{D} .



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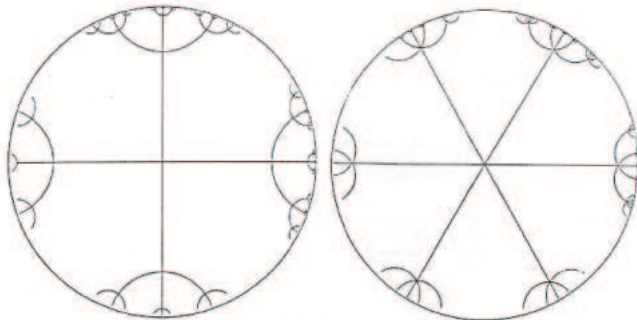
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Examples of Non-Optimal Embeddings

An embedding Φ is called optimal if it satisfies the additional rule:

- The boundary of the disk must be filled up.



Theorem (Cohen & Colonna, 94)

Let $\Phi : T \rightarrow \mathbb{D}$ be a mapping of T which sends neighbors to points in \mathbb{D} of hyperbolic distance r and $r_1 = \cos\left(\frac{\pi}{2t}\right)$. Then

- 1 If $0 < r < r_1$, then Φ is not an embedding.
- 2 If $r_1 \leq r < 1$, then Φ is an embedding and the set L of limit points of $\Phi(T)$ is contained in $\partial\mathbb{D}$.
 - 1 If $r_1 < r < 1$, then L is a set of measure 0.
 - 2 If $r = r_1$, then $L = \partial\mathbb{D}$.

Interpolation Problem

Definition

A sequence $\{z_n\} \subset \mathbb{D}$ is said to be **interpolating** if for every bounded sequence $\{w_n\} \subset \mathbb{C}$ there exists a function f analytic in \mathbb{D} such that $f(z_n) = w_n$.

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In particular, if $w_n = 0$ for all n , does there exist a non-zero analytic function $f : \mathbb{D} \rightarrow \mathbb{C}$ such that $f(z_n) = 0$? For this to occur, the **Blaschke condition** must hold, that is

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If T is a homogeneous tree optimally embedded in \mathbb{D} , and the set of vertices is thought of as an interpolating sequence, then the only function for which f is zero on the vertices is the zero function.

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Theorem (Cohen & Colonna, 94)

The restriction function from the set of bounded analytic functions on \mathbb{D} to bounded functions on T is a monomorphism.

Definition

A function $f : T \rightarrow \mathbb{C}$ is called **Bloch** if

$$\beta_f = \sup_{v \sim w} |f(v) - f(w)| < \infty.$$

The quantity β_f is called the **Bloch constant** of f . The set of Bloch functions on a tree T is denoted by $\mathcal{B}(T)$.

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The function $f(v) = d_T(e, v)$ is an unbounded Bloch function with $\|f\| = 1$.

Mean Value Theorem

Suppose u is harmonic on some domain D in \mathbb{C} and the closed disk centered at a of radius r is contained in D . Then

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) d\theta.$$

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If T is a homogeneous tree of degree d , then a function $f : T \rightarrow \mathbb{C}$ is called **harmonic** if for every vertex v ,

$$f(v) = \frac{1}{d} \sum_{w \sim v} f(w).$$

Definition

If $u(x, y)$ is twice-differentiable on some $\Omega \subset \mathbb{R}^2$, then the **Laplacian** of u is

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

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If we define a Laplacian on T , then it should be 0 on all harmonic functions on T . To do this, we can define Δ on T by

$$\Delta f(v) = \frac{1}{d} \sum_{w \sim v} f(w) - f(v).$$

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 - 1 Can the classical questions be answered about these operators?
 - 2 Can an operator on the Bloch space of T be extended to operators on the Bloch space of \mathbb{D} ?
- 4 Can other spaces of analytic functions be defined for trees?