Isometric Composition Operators on the Bloch Space of $\mathbb{D}$

Joint work with F. Colonna

Agenda

1. Set the Scene
2. The Bloch Space on $\mathbb{D}$
3. Composition Operators on the Bloch Space
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Setting

Let X be a Banach space of analytic functions on $\mathbb{D} = \{|z| < 1\}$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. Define the *composition operator* on X by

$$C_\varphi f = f \circ \varphi.$$

Observation: Clearly for any $\varphi : \mathbb{D} \to \mathbb{D}$, C_φ is linear.

Questions:

- Into what space does C_φ map?
- For what φ does C_φ map into X?
- For what φ is C_φ bounded?
  A linear operator $T : X \to Y$ between normed linear spaces is *bounded* if there exists $M > 0$ such that for all $x \in X$, $||Tx||_Y \leq M\,||x||_X$.
- What is $||C_\varphi||$?
  For a linear operator $T : X \to Y$ between normed linear spaces, $||T|| = \sup\,\{||Tx||_Y : ||x||_X = 1\}$.
- For what φ is C_φ isometric?
  A linear operator $T : X \to X$ on a normed linear space is an *isometry* if $||Tx|| = ||x||$ for all $x \in X$.

We wish to consider answering these questions when X is the Bloch space on $\mathbb{D}$.

The Bloch Space

Definition 1. An analytic function $f : \mathbb{D} \to \mathbb{C}$ is called a *Bloch function* if

$$\beta_f := \sup_{z\in\mathbb{D}} (1 - |z|^2)\, |f'(z)| < \infty.$$

The quantity β_f is called the *Bloch semi-norm*.

Note: The set of Bloch functions becomes a Banach space, called the Bloch space denoted $\mathscr{B}(\mathbb{D})$ of simply $\mathscr{B}$, under the norm

$$||f||_{\mathscr{B}} = |f(0)| + \beta_f.$$

We require the $|f(0)|$ term to distinguish the constant functions.

Theorem 1 (Schwarz-Pick Lemma)**.** *Let $f : \mathbb{D} \to \overline{\mathbb{D}}$ be analytic. Then*

$$(1 - |z|^2)\, |f'(z)| \leq 1 - |f(z)|^2$$

for all $z \in \mathbb{D}$. If f is a conformal automorphism of $\mathbb{D}$, then equality holds for all $z \in \mathbb{D}$, otherwise strict inequality holds.

Proposition 1. *If $f \in H^\infty(\mathbb{D})$, then $\beta_f \leq ||f||_\infty$.*

Proof. If $||f||_\infty = 0$, then $f \equiv 0$. So $\beta_f = 0$. If $0 < ||f||_\infty < \infty$, then define $g(z) = \frac{1}{||f||_\infty} f(z)$. Thus $g : \mathbb{D} \to \overline{\mathbb{D}}$ and

$$\beta_g = \sup_{z\in\mathbb{D}} (1 - |z|^2)\, |g'(z)| \leq \sup_{z\in\mathbb{D}}\ 1 - |g(z)|^2 = 1.$$

Thus $\frac{1}{||f||_\infty}\beta_f = \beta_g \leq 1$. □

Note: $H^\infty \subset \mathscr{B}$. But $\mathscr{B} \not\subset H^\infty$. The function $f(z) = \frac{1}{2}\log(1 - z)$ is Bloch since

$$\beta_f = \frac{1}{2} \sup_{z\in\mathbb{D}} (1 - |z|^2) \left|\frac{1}{1 - z}\right| \leq \frac{1}{2} \sup_{z\in\mathbb{D}} \frac{1 - |z|^2}{1 - |z|} = 1.$$

So the Bloch space contains functions unbounded in the infinity-norm.

Example 1.

1. Polynomials

2. Bounded Functions

   (a) **Möbius Transformations**
   Let $a \in \mathbb{D}$. Define

   $$L_a(z) = \lambda \frac{a-z}{1-\overline{a}z}, \quad |\lambda| = 1.$$

   (b) **Blaschke Products**
   Let $\{a_n\} \subseteq \mathbb{D}$ satisfying $\sum_n 1 - |a_n| < \infty$. Then

   $$B(z) = z^{n_0} \prod_k \left(\frac{\overline{a_k}}{||a_k||} L_{a_k} \right)^{n_k},$$

   where n_0 is the number of occurrences of 0 in $\{a_n\}$ and n_k is the number of occurrences of a_k in $\{a_n\}$.

3. $\log(1-z)$.

Proposition 2. *Let $f \in \mathscr{B}$ and $\varphi : \mathbb{D} \to \mathbb{D}$ analytic. Then $\beta_{f\circ\varphi} \leq \beta_f$. Moreover, if φ is a conformal automorphism, then $\beta_{f\circ\varphi} = \beta_f$, i.e. $\mathscr{B}$ is called* Möbius invariant*.*

Proof.

$$\begin{aligned}
\beta_{f\circ\varphi} &= \sup_{z\in\mathbb{D}} (1-|z|^2)\,|(f\circ\varphi)'(z)| = \sup_{z\in\mathbb{D}} (1-|z|^2)\,|\varphi'(z)|\,|f'(\varphi(z))| \\
&\leq \sup_{z\in\mathbb{D}} (1-|\varphi(z)|^2)\,|f'(\varphi(z))| = \sup_{w\in\varphi(\mathbb{D})} (1-|w|^2)\,|f'(w)| \\
&\leq \sup_{w\in\mathbb{D}} (1-|w|^2)\,|f'(w)| = \beta_f.
\end{aligned}$$

If φ is a conformal automorphism, then both inequalities are equalities. □

Geometric Characterization of Bloch Functions

Definition 2. A *schlicht disk* in the image of f is an open disk $\Delta \subseteq F(\mathbb{D})$ such that there exists a domain $\Omega \subseteq \mathbb{D}$ for which f maps Ω bijectively onto Δ. By $d_f(z_0)$, we mean the radius of the largest schlicht disk in the image of f centered at z_0.

Theorem 2 (Seidel & Walsh, 1942)**.** *The function f is Bloch if and only if the radii of the largest schlicht disk in $f(\mathbb{D})$ is bounded above.*

Corollary 1. *If f is an analytic function on $\mathbb{D}$ such that $f(\mathbb{D})$ has finite area, then f is Bloch.*

Let $\mathcal{D} = \left\{ f : \mathbb{D} \to \mathbb{C} \text{ analytic} \mid \int_{\mathbb{D}} |f'(z)| \; dA(z) < \infty \right\}$, called the *Dirichlet space*. Then $\mathcal{D} \subset \mathscr{B}$.

Theorem 3. *Let $f : \mathbb{D} \to \mathbb{C}$ be analytic. Then the following are equivalent.*

1. *The function f is Bloch.*
2. (Seidel & Walsh, 1942) *The radii of the largest schlicht disk in $f(\mathbb{D})$ is bounded above.*
3. (Pommerenke, 1970) *The family $\{(f \circ \varphi)(z) - (f \circ \varphi)(0) : \varphi \in \mathrm{Aut}(\mathbb{D})\}$ is normal on $\mathbb{D}$.*
4. (Pommerenke, 1970) *There exists a constant $\alpha > 0$ and a univalent analytic function g on $\mathbb{D}$ such that $f(z) = \alpha \log g'(z)$.*
5. (Anderson & Rubel, 1978) *The family*

   $$\left\{ \sum_{j=1}^{n} a_j (f \circ \varphi_j)(z) : n \in \mathbb{N}, \varphi_j \in \mathrm{Aut}(\mathbb{D}), a_j \in \mathbb{C}, \sum_{j=1}^{n} |a_j| \leq 1 \right\}$$

   is normal on $\mathbb{D}$.
6. (Colonna, 1989) *The function f is Lipschitz as a function from $(\mathbb{D}, \rho)$ to $(\mathbb{C}, d)$, where $\rho(z,w) = \frac{1}{2} \log \frac{1+|L_z(w)|}{1-|L_z(w)|}$ is the hyperbolic metric and d is Euclidean distance.*

Composition Operators on the Bloch Space

We now consider Composition Operators on the Bloch Space of $\mathbb{D}$.

Question: For what symbol φ is C_φ a bounded operator on $\mathscr{B}$?
(Madigan & Matheson, 1995) "showed" that if $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic, then C_φ is bounded on $\mathscr{B}$. What they actually showed was that if $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic, then $\beta_{f\circ\varphi} \le \beta_f$ (Proposition 2).

Note: Madigan & Matheson also characterized the compact composition operators on $\mathscr{B}$, but I will not discuss this result.

Thus, the Bloch space has a rich set of symbols which induce bounded composition operators. However, until 2004, no estimates on the norm of C_φ had been established.

(Xiong, 2004) established bounds on the norm of C_φ. In fact, with these bounds, one can actually verify that $||C_\varphi f||_{\mathscr{B}}$ is bounded if φ is an analytic self-map of $\mathbb{D}$.

Notation. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. Define

$$\tau_\varphi(z) = \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\varphi'(z)|, \quad \tau_\varphi^\infty = \sup_{z\in\mathbb{D}} \tau_\varphi(z).$$

Note: By the Schwarz-Pick Lemma, $\tau_\varphi(z) \le 1$, and so $\tau_\varphi^\infty \le 1$.

Theorem 4. *Let φ be a holomorphic mapping of $\mathbb{D}$ into itself, then*

$$\max\left\{1, \frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right\} \le ||C_\varphi|| \le \max\left\{1, \frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|} + \tau_\varphi^\infty\right\}.$$

Corollary 2. *If $\varphi(0) = 0$, then $||C_\varphi|| = 1$.*

This leads to the exploration of isometric composition operators.

Isometric Composition Operators on the Bloch Space

Theorem 5 (Xiong,2004)**.** *Suppose that the composition operator C_φ is isometric on the Bloch space. Then*

1. *φ' is bounded on $\mathbb{D} \Longrightarrow \varphi(z) = e^{i\theta}z$.*
2. *φ is univalent on $\mathbb{D} \Longrightarrow \varphi(z) = e^{i\theta}z$.*

Xiong concluded with the following question:
Does there exist a function φ other than $e^{i\theta}z$ such that C_φ is isometric on the Bloch space?

(Colonna, 2005) answered the question in the affirmative. In fact, Colonna characterized the symbols φ which induce an isometric composition operator on the Bloch space of $\mathbb{D}$.

Theorem 6 (Colonna,2005)**.** *The analytic functions φ from $\mathbb{D}$ into itself which induce an isometric composition operator on the Bloch space are precisely the functions mapping 0 to 0 and having Bloch seminorm equal to 1*

A key ingredient in the proof of the above is the following.

Theorem 7 (Colonna,1989)**.** *Let $\{f_n\}$ be a sequence of Bloch functions converging locally uniformly to some analytic function f. If the sequence $\{f_n\}$ is bounded, then f is Bloch and*

$$\beta_f \leq \liminf_{n\to\infty} \beta_{f_n}.$$

That is, the function $f \mapsto \beta_f$ is lower-semicontinuous on $\mathscr{B}$.

In several papers of Colonna and Cohen and Colonna, we have the following characterization of functions which induce isometric composition operators on the Bloch space.

Theorem 8. *Let φ be an analytic function mapping $\mathbb{D}$ into itself. Then C_φ is an isometry on $\mathscr{B}$ if and only if $\varphi(0) = 0$ and any of the following equivalent conditions hold:*

(a) $\beta_\varphi = 1$.

(b) $B_\varphi := \sup_{z\in\mathbb{D}} \dfrac{(1-|z|^2)\,|f'(z)|}{1-|f(z)|^2} = 1.$ *(Bergman constant)*

(c) *Either* $\varphi \in \mathrm{Aut}(\mathbb{D})$ *or for every* $a \in \mathbb{D}$ *there exists a sequence* $\{z_k\}$ *in* $\mathbb{D}$ *such that* $|z_k| \to 1$, $\varphi(z_k) = a$, *and*

$$\lim_{k\to\infty} \frac{(1-|z_k|^2)\,|\varphi'(z_k)|}{1-|\varphi(z_k)^2|} = 1.$$

(d) *Either* $\varphi \in \mathrm{Aut}(\mathbb{D})$ *or for every* $a \in \mathbb{D}$ *there exists a sequence* $\{z_k\}$ *in* $\mathbb{D}$ *such that* $|z_k| \to 1$, $\varphi(z_k) \to a$, *and*

$$\lim_{k\to\infty} \frac{(1-|z_k|^2)\,|\varphi'(z_k)|}{1-|\varphi(z_k)^2|} = 1.$$

(e) *Either* $\varphi \in \mathrm{Aut}(\mathbb{D})$ *or the zeros of* φ *form an infinite sequence* $\{z_k\}$ *in* $\mathbb{D}$ *such that*

$$\limsup_{k\to\infty} (1-|z_k|^2)\,|\varphi'(z_k)| = 1.$$

(f) *Either* $\varphi \in \mathrm{Aut}(\mathbb{D})$ *or there exists* $\{S_k\}_{k\in\mathbb{N}}$ *in* $\mathrm{Aut}(\mathbb{D})$ *such that* $|S_k(0)| \to 1$ *and* $\{\varphi \circ S_k\}$ *approaches the identity locally uniformly in* $\mathbb{D}$.

(g) *Either* $\varphi \in \mathrm{Aut}(\mathbb{D})$ *or* $\varphi = gB$ *where* g *is a nonvanishing analytic function mapping* $\mathbb{D}$ *into itself or a constant of modulus 1, and* B *is an infinite Blaschke product whose zero set* Z *contains a sequence* $\{z_k\}$ *such that* $|g(z_k)| \to 1$ *and*

$$\lim_{k\to\infty} \prod_{\zeta\in Z,\zeta\neq z_k} \left|\frac{z_k-\zeta}{1-\overline{z_k}\zeta}\right| = 1.$$

Note:

- (d) was noted not by Colonna or Cohen & Colonna, but by (Martin & Vukotić, 2006).
- (g) gives a means on constructing examples of φ that are not in $\mathrm{Aut}(\mathbb{D})$ which induce isometric composition operators on $\mathscr{B}$. Easily constructable examples are Blashcke products whose zero sets are *thin*, ie satisfy

  $$\lim_{k\to\infty} \prod_{j\neq k} \left|\frac{z_k-z_j}{1-\overline{z_k}z_j}\right| = 1.$$

  (Gorkin & Mortini, 2004) showed that sequences in $\mathbb{D}$ satisfying

  $$\lim_{k\to\infty} \frac{1-|z_{k+1}|}{1-|z_k|} = 0$$

are thin. Thus, sequences such as $\{1 - \frac{1}{k!}\}$ and $\{1 - \frac{1}{k^k}\}$ are thin.

Spectrum of Isometric Composition Operator

Having a characterization of the symbols which induce isometric composition operators, we can now classify the spectrum of isometric composition operators on the Bloch space.

Definition 3. Let $T : X \to Y$ be a bounded linear operator between Banach spaces. The *spectrum of* T, denoted $\sigma(T)$ is defined as

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}.$$

Facts:

1. An isometry is necessarily injective.
2. For a bounded operator T, T is invertible if and only if T is bounded below and has dense range.
3. If T is an invertible bounded linear operator, then the inverse is bounded and linear (Inverse Mapping Theorem).
4. The operator norm of an isometry is 1. However, there are operators with norm 1 that are not isometries.
5. The spectrum of an operator T is a generalization of the notion of the eigenvalues of a linear transformation.
6. For an operator T, $\sigma(T) \subseteq \{|z| \le ||T||\}$.
7. The spectrum of an operator is a non-empty compact subset of $\mathbb{C}$.
8. The spectrum of an isometry is a non-empty compact subset of $\overline{\mathbb{D}}$.

Proposition 3 (Allen & Colonna, 07)**.** *Let X be a Banach space and $T : X \to X$ be an isometry. If T is invertible, then $\sigma(T) \subseteq \partial\mathbb{D}$. If T is not invertible, then $\sigma(T) = \overline{\mathbb{D}}$.*

Theorem 9. *Let φ be the symbol of an isometric composition operator on the Bloch space of the unit disk.*

1. *if φ is not a rotation, then $\sigma(C_\varphi) = \overline{\mathbb{D}}$.*
2. *If φ is a rotation with rotation angle $\theta \notin \pi\mathbb{Q}$, then $\sigma(C_\varphi) = \partial\mathbb{D}$.*

3. *if φ is a rotation with angle $\theta \in [0, 2\pi) \cap \pi\mathbb{Q}$, then $\sigma(C_\varphi)$ is the set of m^{th} roots of unity, where m is the smallest positive integer such that $m\theta = 2\pi$.*

Proof. If $\varphi = gB$ as described previously, then C_φ is not invertible. So $\sigma(C_\varphi) = \overline{\mathbb{D}}$.

Let $\varphi(z) = \lambda z$ where $|\lambda| = 1$. The induced composition operator C_φ is an isometry. It is clear that $\varphi^{-1}(z) = \lambda^{-1}z$. This induces the composition operator $C_{\varphi^{-1}} = C_\varphi^{-1}$.

Since C_φ is invertible, $\sigma(C_\varphi) \subseteq \partial\mathbb{D}$. We now want to determine what subset of $\partial\mathbb{D}$ comprises the spectrum of C_φ.

For all $k \in \mathbb{Z}_{\geq 0}$, the function $f(z) = z^k$ is a Bloch function. Also, notice that

$$C_\varphi(f)(z) = f(\varphi(z)) = f(\lambda z) = \lambda^k f(z).$$

So f is an eigenfunction of C_φ with eigenvalue λ^k.

If we define

$$Q_\lambda := \{\lambda^k : k \geq 0\}$$

then

$$\overline{Q_\lambda} \subseteq \sigma(C_\varphi).$$

Case 1. Suppose $\arg(\lambda_j)$ is not a rational multiple of π.

Then Q_λ is dense in $\partial\mathbb{D}$. So $\overline{Q_\lambda} = \partial\mathbb{D}$. Thus $\sigma(C_\varphi) = \partial\mathbb{D}$.

Case 2. Suppose $\arg(\lambda_j)$ is a rational multiple of π. Let m be the order of λ, i.e. the smallest integer such that $\lambda^m = 1$. The group Q_λ generated by λ is of finite order equal to m.

Let $\mu \in \partial\mathbb{D} \setminus Q_\lambda$. We want to show $C_\varphi - \mu I$ is invertible. Since $C_\varphi - \mu I$ is bounded, it suffices to show $C_\varphi - \mu I$ is bijective. This is equivalent to showing for all $g \in \mathcal{B}$ there exists a unique $f \in \mathcal{B}$ such that $C_\varphi(f) - \mu f = g$.

$\varphi^m = \text{id}$. We can now formulate the invertibility of $C_\varphi - \mu I$ in terms of a system of equations. So $C_\varphi - \mu I$ is invertible if and only for every $g \in \mathcal{B}(\mathbb{D}^n)$ the following system has a unique solution $f \in \mathcal{B}$.

$$\begin{array}{lcll} f(\varphi(z)) & - & \mu f(z) & = & g(z) \\ f(\varphi^2(z)) & - & \mu f(\varphi(z)) & = & g(\varphi(z)) \\ & \vdots & & \vdots & \\ f(z) & - & \mu f(\varphi^{\alpha-1}(z)) & = & g(\varphi^{\alpha-1}(z)). \end{array}$$

So, the system of equations has a unique solution if and only if the matrix A in the following system is non-singular:

$$\begin{bmatrix} -\mu & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & & & & \ddots & 1 \\ 1 & 0 & \cdots & \cdots & \cdots & -\mu \end{bmatrix} \begin{bmatrix} f(z) \\ f(\varphi(z)) \\ \vdots \\ \vdots \\ f(\varphi^{\alpha-2}(z)) \\ f(\varphi^{\alpha-1}(z)) \end{bmatrix} = \begin{bmatrix} g(z) \\ g(\varphi(z)) \\ \vdots \\ \vdots \\ g(\varphi^{\alpha-2}(z)) \\ g(\varphi^{\alpha-1}(z)) \end{bmatrix}.$$

The determinant of A is easily calculated as $|A| = (-1)^\alpha(\mu^\alpha - 1)$. This is clearly zero if and only if $\mu^\alpha = 1$ which implies that $\mu \in Q_\lambda$, against the assumption.

Thus, for every $g \in \mathcal{B}$, there exists $f(z) = \sum_{j=1}^{\alpha} a_j g(\varphi^{j-1}(z))$ such that $C_\varphi(f) - \mu f = g$. Since each $g \circ \varphi^k$ is Bloch for all $k = 0, \ldots, \alpha - 1$, f is Bloch also. □