Isometries on the Bloch Space

My Introduction to Operator Theory Research

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Function-Theoretic Operator Theory

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Motivating Example from Linear Algebra

Let A be an $n \times n$ matrix with real entries. We define the operator $L_A : \mathbb{R}^n \to \mathbb{R}^n$ by

$$L_A(u) = Au, \quad u \in \mathbb{R}^n.$$

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L_A is invertible if and only if A is non-singular.

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   T is bounded if there exists $c > 0$ such that $||T(x)|| \leq c||x||, \forall x \in X$.

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$$||T|| = \sup \{||T(x)|| : ||x|| \leq 1\}.$$

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   T is compact if for every bounded sequence $\{x_n\} \subseteq X$, the image $\{T(x_n)\} \subseteq Y$ has compact closure.

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On many spaces, questions 1 and 2 are difficult to answer. Without them, it is difficult to study T in any meaningful way.

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$$\begin{aligned} W_{\psi,\varphi}(f) &= \psi \cdot (f \circ \varphi) \\ &= M_\psi C_\varphi(f). \end{aligned}$$

Outline

1. The Isometry Problem

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2. Isometries on Classical Spaces
3. Introduction to the Bloch Space

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First Solution

The Isometry Problem was first investigated by Banach in 1932, when he proved that the surjective isometries (the invertible isometries) on $C(Q)$, the space of continuous real-valued functions on a compact metric space Q, are of the form

$$Tf = \psi(f \circ \varphi),$$

where $|\psi| \equiv 1$ and φ is a homeomorphism of Q onto itself.

Isometries on Classical Spaces

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Sequence Space ℓ^2

$$\ell^2 = \left\{(a_1, a_2, a_3, \dots) : a_n \in \mathbb{C} \text{ and } \sum |a_n|^2 < \infty\right\}.$$

$$||(a_1, a_2, a_3, \dots)||_2 = \left(\sum |a_n|^2\right)^{1/2}.$$

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- $||S|| = 1$ and is an isometry on ℓ^2.
- $||S^*|| = 1$ but is not an isometry on ℓ^2.

Bergman Space A^p

For $0 < p < \infty$, the Bergman Space A^p consists of analytic functions f on $\mathbb{D}$ such that

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Isometries on A^p [Kolaski, 1982]

For $p \neq 2$, $T : A^p \to A^p$ is a surjective isometry if and only if T has the form

$$Tf = \lambda(\varphi')^{2/p}(f \circ \varphi)$$

where φ is an automorphism of $\mathbb{D}$ and λ is a constant of modulus 1.

Isometries on Classical Spaces

Hardy Space H^p

For $0 < p < \infty$, the Hardy Space H^p consists of analytic functions f on the open unit disk $\mathbb{D}$ such that

$$||f||_p = \left(\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p \, d\theta \right)^{1/p} < \infty.$$

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Isometries on H^p [Forelli, 1964]

For $p \neq 2$, $T : H^p \to H^p$ is an isometry if and only if

$$Tf = \psi(f \circ \varphi),$$

for some ψ and φ.

Introduction to the Bloch Space

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Definition

An analytic function $f : \mathbb{D} \to \mathbb{C}$ *is said to be Bloch provided*

$$\beta_f := \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| f'(z) \right| < \infty.$$

The Bloch space, defined as $\mathcal{B} = \{f : \mathbb{D} \to \mathbb{C} \text{ analytic} \mid \beta_f < \infty\}$, *is a Banach space under the norm* $||f||_{\mathcal{B}} = |f(0)| + \beta_f$.

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Schwarz-Pick Lemma

Let $f : \mathbb{D} \to \overline{\mathbb{D}}$ be analytic. Then for $z \in \mathbb{D}$,

$$(1 - |z|^2) \left| f'(z) \right| \leq 1 - |f(z)|^2 .$$

If $f(z)$ is a conformal automorphism of $\mathbb{D}$, then equality holds; otherwise the inequality is strict for all $z \in \mathbb{D}$.

Important Consequences of the Schwarz-Pick Lemma

Proposition

For $f \in \mathcal{B}$ and φ analytic from $\mathbb{D}$ into $\mathbb{D}$, then

$$\beta_{f \circ \varphi} \leq \beta_f.$$

Moreover, if φ is a conformal automorphism of $\mathbb{D}$, then equality holds; $\mathcal{B}$ is Möbius Invariant.

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The set of bounded analytic functions on $\mathbb{D}$ is denoted $H^\infty(\mathbb{D})$, and is a Banach space under the norm $||f||_\infty = \sup\limits_{z\in\mathbb{D}} |f(z)|$.

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Corollary

$H^\infty(\mathbb{D})$ is contained in $\mathcal{B}$.

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   $$\varphi_a(z) = \lambda \frac{a - z}{1 - \overline{a}z}.$$
   2. Let $\{a_n\}$ be a sequence in $\mathbb{D}$ satisfying the Blaschke condition of $\sum_n (1 - |a_n|) < \infty$. We define the Blaschke product as
   $$B(z) = z^m \prod_{a_k \neq 0} \left(\frac{\overline{a_k}}{|a_k|} \frac{a_k - z}{1 - \overline{a_k}z} \right),$$
   where $m \in \mathbb{Z}_+$ is the number of times 0 appears in the sequence.

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   where $m \in \mathbb{Z}_+$ is the number of times 0 appears in the sequence.
3. $f(z) = \operatorname{Log}\left(\frac{1+z}{1-z}\right)$.

Isometries on the Bloch Space – *The Past*

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Theorem [Cima & Wogen, 1980]

Let $T : \mathcal{B}_* \to \mathcal{B}_*$ be an onto isometry. Then there exists a conformal automorphism φ of $\mathbb{D}$ and a $\lambda \in \partial\mathbb{D}$ so that

$$T(f) = \lambda(f \circ \varphi - f(\varphi(0))),$$

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Such an operator is called a compression of the composition operator C_φ.

Isometries on the Bloch Space – *The Present*

Trend in the Study of Isometries on Function Spaces

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3. Using the estimates, determine a class of symbols which induce an isometric operator.
4. Determine a complete characterization of the symbols of isometric operators.

Isometric Composition Operators

Boundedness

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Norm Estimates [Xiong, 2004]

If φ is any analytic self-map of $\mathbb{D}$, then

$$\max\left\{1, \frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right\} \leq ||C_\varphi|| \leq \max\left\{1, \frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|} + \tau_\varphi^\infty\right\}$$

where

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If $\varphi(0) = 0$ and $\beta_\varphi = 1$, then $||C_\varphi|| = 1$. In fact, C_φ is an isometry as well.

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Symbols which induce isometry [Xiong, 2004]

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Characterization [Colonna, 2005]

C_φ is an isometry on the Bloch space if and only if $\varphi(0) = 0$ and $\beta_\varphi = 1$. Moreover, φ is a rotation or $\varphi = gB$ where $g : \mathbb{D} \to \overline{\mathbb{D}}$ is a non-vanishing analytic function and B is a Blaschke product whose zeros form an infinite sequence $\{z_n\}$ containing 0 and an infinite subsequence $\{z_{n_j}\}$ such that $\left|g(z_{n_j})\right| \to 1$ and

$$\lim_{j\to\infty} \prod_{k\neq n_j} \left| \frac{z_{n_j} - z_k}{1 - \overline{z_{n_j}} z_k} \right| = 1.$$

Isometric Multiplication Operators

Boundedness [Brown and Shields, 1991]

M_ψ is bounded on the Bloch space if and only if $\psi \in H^\infty(\mathbb{D})$ and

$$\left|\psi'(z)\right| = O\left(\frac{1}{(1-|z|)\log\frac{1}{1-|z|}}\right).$$

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Norm Estimates [A. and Colonna, 2009]

If M_ψ is bounded on the Bloch space, then

$$\max\left\{||\psi||_{\mathcal{B}}\,, ||\psi||_\infty\right\} \leq ||M_\psi|| \leq \max\left\{||\psi||_{\mathcal{B}}\,, ||\psi||_\infty + \sigma_\psi^\infty\right\}$$

where

$$\sigma_\psi^\infty = \sup_{z\in\mathbb{D}} \frac{1}{2}(1-|z|^2)\left|\psi'(z)\right|\log\frac{1+|z|}{1-|z|}.$$

Isometric Multiplication Operators

Symbols which induce isometry

Let $\psi(z) = \lambda$ where $|\lambda| = 1$, then M_ψ is an isometry on the Bloch space.

$$||M_\psi f||_{\mathcal{B}} = ||\psi f||_{\mathcal{B}} = |\psi(0)|\,|f(0)| + \beta_{\psi f} = ||f||_{\mathcal{B}}\,.$$

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Characterization [A. and Colonna, 2009]

M_ψ is an isometry on the Bloch space if and only if ψ is a constant function of modulus 1.

Isometric Weighted Composition Operators

Boundedness [Ohno and Zhou, 2001]

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Problematic Example

Let $\psi(z) = \log\frac{2}{1-z}$ and $\varphi(z) = \frac{1-z}{2}$. Then $W_{\psi,\varphi}$ is bounded on the Bloch space, while M_ψ is not bounded.

Isometric Weighted Composition Operators

Norm Estimates [A. and Colonna, 2009]

If $W_{\psi,\varphi}$ is bounded on the Bloch space, then

$$\max\left\{||\psi||_{\mathcal{B}}, \frac{1}{2}|\psi(0)|\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right\} \leq ||W_{\psi,\varphi}|| \leq$$
$$\max\left\{||\psi||_{\mathcal{B}}, \frac{1}{2}|\psi(0)|\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|} + \tau^{\infty}_{\psi,\varphi} + \sigma^{\infty}_{\psi,\varphi}\right\}$$

where

$$\tau^{\infty}_{\psi,\varphi} = \sup_{z\in\mathbb{D}} \frac{1-|z|^2}{1-|\varphi(z)|^2}|\psi(z)|\left|\varphi'(z)\right|, \text{ and}$$
$$\sigma^{\infty}_{\psi,\varphi} = \sup_{z\in\mathbb{D}} \frac{1}{2}(1-|z|^2)\left|\psi'(z)\right|\log\frac{1+|\varphi(z)|}{1-|\varphi(z)|},$$

Isometric Weighted Composition Operators

Symbols which induce isometry

If ψ induces an isometric multiplication operator M_ψ and φ induces an isometric composition operator C_φ, then $W_{\psi,\varphi}$ is an isometry on the Bloch space.

$$||W_{\psi,\varphi}f||_{\mathcal{B}} = ||M_\psi(C_\varphi f)||_{\mathcal{B}} = ||C_\varphi f||_{\mathcal{B}} = ||f||_{\mathcal{B}}\,.$$

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Characterization

Isometries on the Bloch Space – *The Future*

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4. Develop characterizations of isometries for the Bloch space on discrete structures.

Thank You For Coming!