Classical and Modern Operator Theory on the Bloch Space

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Introduction

A Motivating Example from Linear Algebra

Let A be an $n \times n$ matrix with real entries. We define the operator $L_A : \mathbb{R}^n \to \mathbb{R}^n$ by

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- The study of operators with symbols is called Function-Theoretic Operator Theory.
- The goal is to link properties of the symbol with properties of the operator.
- The perfect marriage of (complex) Function Theory and Operator Theory.

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$$\begin{aligned} W_{\psi,\varphi} f &= \psi(f \circ \varphi) \\ &= M_\psi C_\varphi f. \end{aligned}$$

- In one variable, the open unit disk $\mathbb{D}$ is the most common choice for a domain when studying the function theory of spaces of analytic functions and operators on such spaces.

Operator Theory in Several Complex Variables

- In one variable, the open unit disk $\mathbb{D}$ is the most common choice for a domain when studying the function theory of spaces of analytic functions and operators on such spaces.
- Two common generalizations of $\mathbb{D}$ in higher dimensions are:
  1. The Unit Ball
  $$\mathbb{B}_n = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^2 < 1 \right\}.$$
  2. The Unit Polydisk
  $$\mathbb{D}^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| < 1, j = 1, \ldots, n\}.$$

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- Typically, a researcher will choose one of these two domains on which to study operator theory.

Unified Operator Theory

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- So by studying spaces of holomorphic functions on bounded homogeneous domains, we bring together the study of spaces on the unit ball and unit polydisk.
- Each bounded homogeneous domain is endowed with a metric known as the Bergman metric, which is invariant under automorphisms.

Definition

A domain $D \subseteq \mathbb{C}^n$ is called symmetric at $z_0 \in D$ if there exists an automorphism φ of D such that $\varphi \circ \varphi$ is the identity map on D and z_0 is an isolated fixed point of φ. The domain D is called symmetric if it is symmetric at each $z \in D$.

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- Every symmetric domain is homogeneous, and every homogeneous domain that is symmetric at a point is symmetric.
- The unit ball and unit polydisk are symmetric domains.
- Every symmetric domain can be written as a product of irreducible symmetric domains.

- D will denote a bounded homogeneous domain in $\mathbb{C}^n$, unless otherwise specified.

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- ∂D denotes the boundary of D.
- $H^\infty(D)$ denotes the Banach space of bounded holomorphic functions on domain D under the infinity norm $||f||_\infty = \sup_{z \in D} |f(z)|$.

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  - $B_\varphi = \sup_{z \in D} B_\varphi(z)$.

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- The spectrum of $T : X \to X$ is

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\},$$

where I is the identity operator on X.

The Bloch Space

Bloch Space on the Unit Disk

Definition

A function $f \in H(\mathbb{D})$ is called Bloch if

$$\beta_f := \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| f'(z) \right| < \infty.$$

The Bloch space on $\mathbb{D}$, denoted by $\mathcal{B}(\mathbb{D})$, is a Banach space under the norm

$$||f||_{\mathcal{B}} = |f(0)| + \beta_f.$$

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Examples of Bloch Functions

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- $\operatorname{Log} \dfrac{1+z}{1-z}$.

Bloch Space on a Bounded Homogeneous Domain

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The Bloch space on D, denoted by $\mathcal{B}(D)$, is a Banach space under the norm

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Polynomials and bounded holomorphic functions are Bloch.

Multiplication Operators on $\mathcal{B}$

Previous Research on $\mathcal{B}(\mathbb{D})$

Theorem (Arazy, 82)

For $\psi \in H(\mathbb{D})$, M_ψ is bounded if and only if $\psi \in H^\infty(\mathbb{D})$ and

$$\sup_{z\in\mathbb{D}} \frac{1}{2}(1-|z|^2)\left|\psi'(z)\right| \log\frac{1+|z|}{1-|z|} < \infty.$$

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Theorem (Ohno and Zhao, 01)

For $\psi \in H(\mathbb{D})$, M_ψ is compact if and only if $\psi \equiv 0$.

Previous Research on $\mathcal{B}(\mathbb{B}_n)$

Considering the Bloch space under the norm

$$||f|| = |f(0)| + \sup_{z \in \mathbb{B}_n} (1 - ||z||^2) \, ||\nabla(f)(z)|| \, ,$$

Zhu characterized the bounded multiplication operators on $\mathcal{B}(\mathbb{B}_n)$.

Theorem (Zhu, 04)

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3. Develop a unified operator theory for M_ψ in higher dimensions.
   - Characterize boundedness.
   - Determine operator norm estimates.
   - Determine the spectrum.
   - Characterize compactness.
   - Characterize the isometries.

Multiplication Operators on $\mathcal{B}(\mathbb{D})$

Theorem

Suppose ψ is the symbol of a bounded multiplication operator M_ψ. Then

$$\max\{||\psi||_{\mathcal{B}}, ||\psi||_\infty\} \leq ||M_\psi|| \leq \max\{||\psi||_{\mathcal{B}}, ||\psi||_\infty + \sigma_\psi\},$$

where

$$\sigma_\psi = \sup_{z \in \mathbb{D}} \frac{1}{2}(1 - |z|^2) \left|\psi'(z)\right| \log \frac{1 + |z|}{1 - |z|}.$$

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Theorem

The multiplication operator M_ψ is an isometry if and only if ψ is a constant function of modulus 1.

Generalizing to Higher Dimensions

Theorem (Zhu, 01)

For $z \in \mathbb{B}_n$,

$$\begin{aligned}\rho(0,z) &= \frac{1}{2}\log\frac{1+\|z\|}{1-\|z\|}\\ &= \sup\{|f(z)| : f \in \mathcal{B}(\mathbb{B}_n), f(0)=0, \text{ and } \|f\|_{\mathcal{B}} \leq 1\}\end{aligned}$$

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Definition

Let D be a bounded homogeneous domain. For $z \in D$, let

$$\omega(z) = \sup\{|f(z)| : f \in \mathcal{B}(D), f(0)=0, \text{ and } ||f||_{\mathcal{B}} \leq 1\},$$

Definition

For $\psi \in H(D)$ and $z \in D$, let $\sigma_\psi(z) = \omega(z)Q_\psi(z)$ and $\sigma_\psi = \sup\{\sigma_\psi(z) : z \in D\}$.

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$$\max\{\|\psi\|_{\mathcal{B}}, \|\psi\|_\infty\} \leq \|M_\psi\| \leq \max\{\|\psi\|_{\mathcal{B}}, \|\psi\|_\infty + \sigma_\psi\}.$$

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Let $\psi \in H(D)$ such that M_ψ is bounded on $\mathcal{B}(D)$. Then $\sigma(M_\psi) = \overline{\psi(D)}$.

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Let $\psi \in H(D)$. Then M_ψ is compact on $\mathcal{B}(D)$ if and only if ψ is identically zero.

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Theorem

Let $D = D_1 \times \cdots \times D_k$ be a bounded symmetric domain with $D_j \neq \mathbb{D}$ for all $j \in \{1, \ldots, k\}$, and $\psi \in H(D)$. Then M_ψ is an isometry on $\mathcal{B}(D)$ if and only if ψ is a constant function of modulus one.

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Theorem

Let $\psi \in H(\mathbb{D}^n)$. Then M_ψ is an isometry on $\mathcal{B}(\mathbb{D}^n)$ if and only if ψ is a constant function of modulus one.

Remark

We have a characterization of the isometric M_ψ for bounded symmetric domains in which all the factors are $\mathbb{D}$ and none of the factors are $\mathbb{D}$. We, however, do not have a characterization for such domains where only some of the factors are $\mathbb{D}$.

Weighted Composition Operators on $\mathcal{B}$

Definition

For $\psi \in H(\mathbb{D})$, φ an analytic self-map of $\mathbb{D}$, and $z \in \mathbb{D}$, let

$$s_{\psi,\varphi}(z) = \frac{1}{2}(1-|z|^2)\left|\psi'(z)\right| \log\frac{2}{1-|\varphi(z)|^2}, \qquad s_{\psi,\varphi} = \sup_{z\in\mathbb{D}} s_{\psi,\varphi}(z),$$

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Previous Research

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Theorem (Ohno and Zhao, 01)

Let $\psi \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then

1. *$W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{D})$ if and only if $s_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite.*
2. *$W_{\psi,\varphi}$ is compact on $\mathcal{B}(\mathbb{D})$ if and only if $W_{\psi,\varphi}$ is bounded and*

$$\lim_{|\varphi(z)|\to 1} s_{\psi,\varphi}(z) = \lim_{|\varphi(z)|\to 1} \tau_{\psi,\varphi}(z) = 0.$$

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Theorem (Zhou and Chen, 05)

Let $\psi \in H(\mathbb{B}_n)$ and φ be a holomorphic self-map of $\mathbb{B}_n$. Then $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{B}_n)$ if and only if

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   - Characterize boundedness.
   - Determine operator norm estimates.
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Weighted Composition Operators on $\mathcal{B}(\mathbb{D})$

Definition

For $\psi \in H(\mathbb{D})$, φ an analytic self-map of $\mathbb{D}$, and $z \in \mathbb{D}$, let

$$\sigma_{\psi,\varphi}(z) = \frac{1}{2}(1-|z|^2)\left|\psi'(z)\right| \log \frac{1+|\varphi(z)|}{1-|\varphi(z)|}, \quad \sigma_{\psi,\varphi} = \sup_{z\in\mathbb{D}} \sigma_{\psi,\varphi}(z).$$

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Theorem

Let $\psi \in H(\mathbb{D})$ and φ an analytic self-map of $\mathbb{D}$ which induce a bounded weighted composition operator. Then

1. $||W_{\psi,\varphi}|| \geq \max\left\{||\psi||_{\mathcal{B}}, \frac{1}{2}|\psi(0)| \log \frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right\}$.
2. $||W_{\psi,\varphi}|| \leq \max\left\{||\psi||_{\mathcal{B}}, \frac{1}{2}|\psi(0)| \log \frac{1+|\varphi(0)|}{1-|\varphi(0)|} + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi}\right\}$.

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Let $\psi \in H(\mathbb{D})$ and φ an analytic self-map of $\mathbb{D}$. Then $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{D})$ if and only if $\psi \in \mathcal{B}(\mathbb{D})$, and $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite.

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Let $\psi \in H(\mathbb{D})$ *and* φ *be an analytic self-map of* $\mathbb{D}$*. Then* $W_{\psi,\varphi}$ *is compact on* $\mathcal{B}(\mathbb{D})$ *if and only if* $W_{\psi,\varphi}$ *is bounded and*

$$\lim_{|\varphi(z)|\to 1} \sigma_{\psi,\varphi}(z) = \lim_{|\varphi(z)|\to 1} \tau_{\psi,\varphi}(z) = 0.$$

Generalizing to Higher Dimensions

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For φ a holomorphic self-map of D, and $z \in D$, let

$$T_\varphi(z) = \sup\{Q_{f\circ\varphi}(z) : f \in \mathcal{B}(D), \beta_f \leq 1\}.$$

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Foundation for a Unified Theory for $W_{\psi,\varphi}$

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Let φ a holomorphic self-map of D. If $\psi \in \mathcal{B}(D)$, and $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite, then $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(D)$.

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Conjecture

Let $\psi \in H(D)$ and φ a holomorphic self-map of D. Then $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(D)$ if and only if $\psi \in \mathcal{B}(D)$, and $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite.

Foundation for a Unified Theory for $W_{\psi,\varphi}$

Theorem

Let D be a bounded homogeneous domain. If $\psi \in H(D)$ and φ a holomorphic self-map of D induce a bounded weighted composition operator $W_{\psi,\varphi}$ on $\mathcal{B}(D)$, then

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Let $\psi \in H(D)$ and φ a holomorphic self-map of D. If

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Let $\psi \in H(\mathbb{B}_n)$ and φ be a holomorphic self-map of $\mathbb{B}_n$. Then $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{B}_n)$ if and only if $\psi \in \mathcal{B}(\mathbb{B}_n)$ and $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite.

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Further Directions

Unified Theory

- Complete the unification for $W_{\psi,\varphi}$ on the Bloch space.

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- Unify the operator theory for M_ψ, C_φ, and $W_{\psi,\varphi}$ on the Hardy space defined on a bounded symmetric domain.

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Thank You For Coming!