Suppose we are studying the nonlinear system

$$\frac{dx}{dt} = f(x, y) \qquad \frac{dy}{dt} = g(x, y). \tag{4}$$

We seek a pair of functions $x = x(t)$ and $y = y(t)$ that simultaneously satisfy both equations. We can think of these equations as governing the motion of a particle in the xy-plane as a function of time t.

A special kind of solution is called an **equilibrium solution**. These are solutions which remain constant for all time so that solution stays at the same **equilibrium point** in the xy-plane for all time. To find such a solution we require that $\frac{dx}{dt} = \frac{dy}{dt} = 0$ (no change in x or y). Thus we have to simultaneously solve the equations

$$f(x, y) = 0, \qquad g(x, y) = 0. \tag{5}$$

A solution (x_0, y_0) to (5) is called an equilibrium solution or an equilibrium point.

Example: Find equilibrium solutions for the mutual competition model of rabbits versus sheep (both are competing for the same grass, R and S are measured in the hundreds):

$$\begin{aligned} \frac{dR}{dt} &= 3R - R^2 - 2RS \\ \frac{dS}{dt} &= 2S - S^2 - RS \end{aligned}$$

For nonlinear equations it can be difficult to predict the behavior of solutions (analytically) except in special cases, however for solutions that close to equilibrium points we can make some progress. Suppose $(x(t), y(t))$ is near an equilibrium solution (x_0, y_0). Define the difference between the equilibrium solution and the nearby solution as

$$\Delta x(t) = x(t) - x_0, \quad \Delta y(t) = y(t) - y_0$$

so that

$$x = x_0 + \Delta x, \quad y = y_0 + \Delta y. \tag{6}$$

We've left off the (t) symbols for simplicity. Notice that Δx and Δy depend on t and represent the displacement from the equilibrium solution which constant and independent of t.

Now we substitute (6) into (4) to get

$$\begin{aligned} \frac{d(x_0+\Delta x)}{dt} &= f(x_0+\Delta x, y_0+\Delta y) && (7)\\ \frac{d(y_0+\Delta y)}{dt} &= g(x_0+\Delta x, y_0+\Delta y) && (8)\\ & && (9) \end{aligned}$$

On the left hand side of these equations we use the fact that $\frac{dx_0}{dt} = \frac{dy_0}{dt} = 0$ to write

$$\frac{d(x_0+\Delta x)}{dt} = \frac{dx_0}{dt} + \frac{d\Delta x}{dt} = \frac{d\Delta x}{dt} \tag{10}$$

and

$$\frac{d(y_0+\Delta y)}{dt} = \frac{dy_0}{dt} + \frac{d\Delta y}{dt} = \frac{d\Delta y}{dt} \tag{11}$$

On the right hand side we'll use Taylor Series expansion about the point (x_0, y_0) to write

$$f(x_0+\Delta x, y_0+\Delta y) = f(x_0,y_0) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \text{ higher order terms} \tag{12}$$

The expansion for g is similar. The higher order terms involve higher derivatives and higher powers of Δx and Δy. Notice that if $(x(t), y(t))$ is close to an equilibrium solution (x_0, y_0) then $(\Delta x, \Delta y)$ will both be small and the higher order terms become neglible. Also, $f(x_0, y_0) = g(x_0, y_0) = 0$. So after expanding the right hand side and dropping the higher order terms we have the following system of differential equations governing the evolution of the displacements Δx and Δy from the equilibrium.

$$\begin{aligned} \frac{d\Delta x}{dt} &= f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y && (13)\\ \frac{d\Delta y}{dt} &= g_x(x_0,y_0)\Delta x + g_y(x_0,y_0)\Delta y && (14)\\ & && (15) \end{aligned}$$

The partial derivatives are all evaluated at the equilibrium solution so each of those terms is just a number multiplying Δx or Δy, thus this is a linear system of ODE's:

$$\frac{d}{dt}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \tag{16}$$

The matrix in (16) is called the Jacobian matrix:

$$J(x,y) = \begin{bmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{bmatrix}$$

Example: Find the Jacobian matrix and evaluate it at each of the equilibrium solutions.

$$\begin{aligned} \frac{dR}{dt} &= 3R - R^2 - 2RS \\ \frac{dS}{dt} &= 2S - S^2 - RS \end{aligned}$$

Near the equilibrium at $(3, 0)$ we have

$$\frac{d}{dt}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \tag{17}$$

The general behavior of the (small) displacements Δx and Δy is determined by the eigenvalues of Jacobian matrix, in this case $\lambda_1 = -3, \lambda_2 = -1$, so both eigenvalues are negative. Solutions to this linear system will have the form

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = c_1 e^{-3t}\vec{v}_1 + c_2 e^{-t}\vec{v}_2 \tag{18}$$

where $\vec{v}_1$ and $\vec{v}_2$ are the corresponding eigenvectors of the Jacobian matrix. For our purposes we just want to predict the general behavior or **stability** of the solutions near the equilibrium so it is enough to know that the eigenvalues are negative and that the displacements will decay to zero. So solutions that start near this equilibrium will decay toward this equilibrium - it is called a sink or a stable node.

For a review of the general behavior of linear systems, eigenvalues, and the phase planes, see Section 9.1 in your textbook. Table 9.1.1 on page 492 may be particularly useful.

Another way to characterize these different sorts of solutions for 2D linear systems is to use the Trace-Determinant Plane: