DUE: MONDAY, DECEMBER 5, 2011

And NUH is the letter I use to spell Nutches,
Who live in small caves, known as Nitches, for hutches.
These Nutches have troubles, the biggest of which is
The fact there are many more Nutches than Nitches.
Each Nutch in a Nitch knows that some other Nutch
Would like to move into his Nitch very much.
So each Nutch in a Nitch has to watch that small Nitch
Or Nutches who haven't got Nitches will snitch.

-Dr. Seuss, *On Beyond Zebra* (1955)

In this project, you will study the one-parameter family of nonlinear, first order system consisting of predator-prey equations. The family is

$$\frac{dx}{dt} = 9x - \alpha x^2 - 3xy$$
$$\frac{dy}{dt} = -2y + xy$$

where $\alpha \geq 0$ is a parameter. In other words, for different values of α we have different systems. The variable x is the population (in some scaled units) of prey, and y is the population of predators. For a given value of α, we want to understand what happens to these populations as $t \to \infty$.
You should investigate the phase portraits of these systems for various values of α in the interval $0 \leq \alpha \leq 5$. To get started, you might want to try $\alpha = 0, 1, 2, 3, 4$, and 5. (Use the phase plane program linked from our HW page.) Think about what the phase portrait means in terms of the evolution of the x and y populations. Where are the equilibrium solutions? What does the Jacobian tell you about their types? What happens to a typical solution curve? Also, consider the behavior of the special solutions where either $x = 0$ or $y = 0$ (solution curves lying on the x- or y-axes).
Determine the bifurcation values of α–that is, the values of α where nearby α's lead to "different" behaviors in the phase portrait. For example $\alpha = 0$ is a bifurcation value because for $\alpha = 0$, the long-tem behavior of the populations is dramatically different than the long-term behavior of the populations if α is slightly positive. The process of finding the equilibrium solutions and classifying for the equation above should suggest bifurcation values. Find all of them.

Your report: After you have determined all of the bifurcation values for α in the interval $0 \leq \alpha \leq 5$, study enough specific values of α to be able to discuss all of the various population evolution scenarios for these systems. In your report, you should describe these scenarios using the phase portraits. Your report should include:

1. A brief discussion of the significance of the various terms in the system. For example, what does $9x$ represent? Why is it positive? What does the $3xy$ term represent? etc.

2. A discussion of all bifurcations including the bifurcation at $\alpha = 0$. For example, a bifurcation occurs between $\alpha = 3$ and $\alpha = 5$. What does this bifurcation mean for the predator population?

Address the following questions in the form of a short essay (I **do not** want you to say the answer question 1 is . . .), and support your assertions with selected illustrations. (Please remember that although one good illustration may be worth 1000 words, 1000 illustrations are usually worth nothing.)

Have questions? Feel free to stop by my office (1026 Cowley Hall) or contact me by email.
Hints

1. Initially, the phase portraits at alpha = 0, 1, 2, 3, 4, 5 are only used to help you see that the dynamics are changing as α changes. Those are not the bifurcation points. You must do some work by hand to find the exact bifurcation points (there are 3 of them in total).

2. Both populations must be non-negative. This condition will restrict the number of the euilibrium solutions leading to two of the bifurcation points.

3. Find the equlibrium solutions as a function of α.

4. Break your analysis up into cases depending on which side of the bifurcation point(s) α lies on.

5. For each case, determine the Jacobian at each equilibrium (the Jacobian will be dependent on α as well) and analyze the type of equilibrium. You do not need to calculate any eigenvectors.

6. Provide a phase portrait (for example the appropriate one from comment 1. above) and description (using the predator-prey language) for each case.