A **differential equation** is an equation that relates a dependent variable (function, e.g. $y(t)$) and its derivatives ($y'(t), y''(t)$, etc.) and the indepent variables (in this case just t). Here are some examples:

Example: Exponential growth ($k > 0$) or exponential decay ($k < 0$):

$$\frac{dy}{dt} = ky$$

To give a more specific example, let $y(t)$ represent dollars in year t or the size of population at time t and let $k = 0.03$:

$$\frac{dy}{dt} = 0.03y$$

What does this mean?

What is the solution?

Verifying a solution.

Initial Value Problem.

Example: The logistic population equation, $k > 0, M > 0$ (constants)

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right)$$

What does the new term do?

specific example:

$$\frac{dy}{dt} = 0.05y\left(1 - \frac{y}{100}\right)$$

while it is possible to find an analytic solution to this equation, it really would not tell us much, instead we'll use a

qualitative approach: using graphics and properties of the equation to predict the approximate the behavior of solutions.

autonomous equations:

$\frac{dy}{dt} = f(y)$, that is, the right side is a function of **only** the dependent variable.

We are used to the opposite in first semester calculus:

Example:

$$\frac{dy}{dt} = 2t - \cos(t)$$

A more general (explicit) D.E. is:

$$\frac{dy}{dt} = f(t, y)$$

e.g.

$$\frac{dy}{dt} = t^4 y$$

This last equation is an example of a **separable** equation. These can be broken up into separate integrals in the two variables:

Example: (BDH page 34, #28)

$$\frac{dy}{dt} = \frac{t}{y - t^2 y}, y(0) = 4.$$

Homework: (BDH)
1.1: 3, 5, 10, 11, 15
1.2: 2, 5, 7, 11, 23, 25, 31, 33, 35

For much of this work we'll use the internet program DFIELD available at `http://math.rice.edu/~dfield/dfpp.html`.

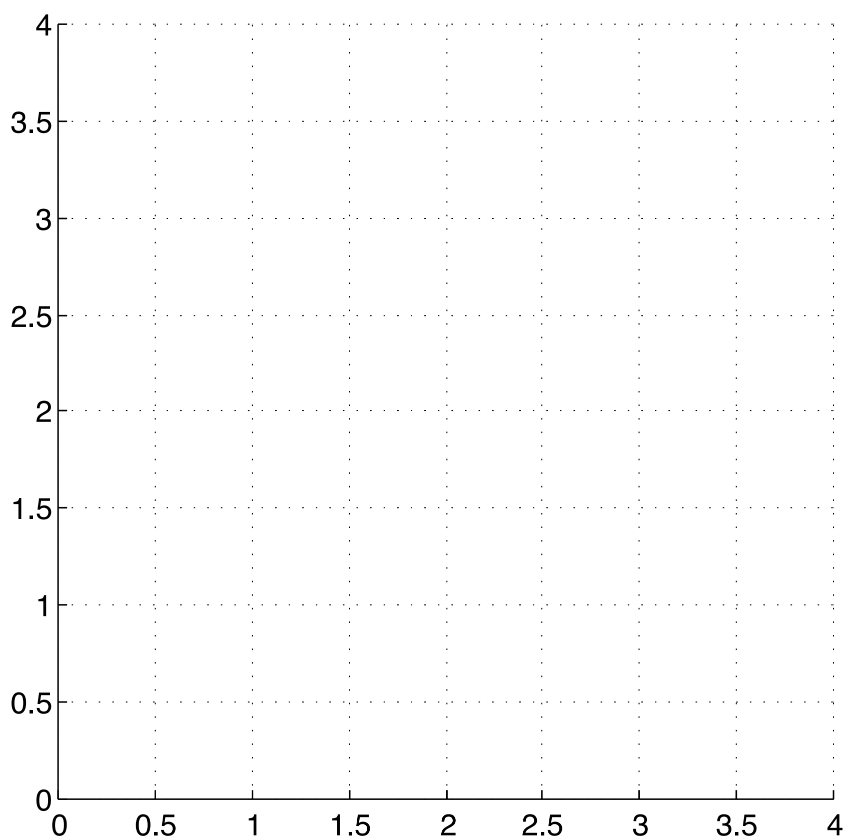
Section 1.3: Qualitative Technique, Slope Field (Direction Field)

Consider the differential equation

$$\frac{dy}{dt} = f(t, y)$$

One view of this is that function $f(t, y)$ can be used to determine the slope of the curve $y(t)$ at any point (t, y). Just plug the desired values for t and y into the function $f(t, y)$ and since $y'(t) = \frac{dy}{dt} = f(t, y)$ we now know the slope of $y(t)$ at the point (t, y). This is the main idea behind slope fields.

At a lattice of points in the $t - y$ plane evaluate the differential equation to find the slope of the solution, dy/dt. Indicate the slope graphically with a dash at that slope through the point (t, y). These dashes are tangent to the solutions.
Example: $\frac{dy}{dt} = (y^2 + 1)t$



Doing this sort of thing by hand gets quite tedious (though checking the slope at a point or two is really not difficult), so we turn to technology.

DFIELD actually uses arrows instead of dashes so it is easier to read the direction of the solution. We'll do in class demos of DFIELD, but if you're not in class then use the link above to get to DFIELD and input some of the equations and graphing windows below to explore. Be sure to also click in the "Direction Field Window" to see the solutions themselves.

Demo Example: $\frac{dy}{dt} = (y^2+1)t, (t,y) \in [0,4] \times [0,4]$. Experiment with other graphing windows. Note that solutions are graphed forward and backward in time.

The solutions grow faster than exponential and seem to blow up. Do they exist for all time?

Demo Example: The logistic equation

$\frac{dy}{dt} = 0.05y\left(1 - \frac{y}{100}\right), (t,y) \in [0,100] \times [-10,120]$

Do the features agree with our phase line analysis from last time? This is an autonomous equation, what is special about the direction field?

What if the DE depends only on the independent variable?

Demo Example: $\frac{dy}{dx} = x(x-1)(x-2)$

HW - 1.3 (BDH): 3, 4, 5 (use DFIELD for tech. bit), 7-10 (DFIELD), 12, 15 (no tech.), 17

1.6: Autonomous equations: Equilibria and the phase line

$$\frac{dy}{dt} = f(y)$$

An equibrium point is a point (value) of y_0 such that $f(y_0) = 0$. This corresponds to a solution that is constant in time $y(t) = y_0$ for all t. While we haven't discussed this, under mild conditions, the solutions to DEs do not cross (the solution through any point is **unique**), so no solutions will cross the constant equilibrium solutions. (This will make more sense in a minute..)

Equilibrium solutions can have different characteristics as we saw in the logistic equation example and we'll see further in the example below:

Example: (# 8 page 91 BDH)

$$\frac{dw}{dt} = 3w^3 - 12w^2$$

Find the equilibrium points, classify them as sources, sinks, or nodes. Use the phase diagram to sketch graphs of several solutions.

HW - 1.6 (BDH): 1, 3, 7, 9, 13, 15, 19, 21, 22-27 (a few), 31, 35, 43

1.6 Autonomous Equations continued

A nice theorem for analyzing the stability of equilibria of autonomous equations:

$$\frac{dy}{dt} = f(y).$$

Theorem: (Linearization) Let y_0 be an equilbrium point of the autonomous equation ($f(y_0) = 0$). If

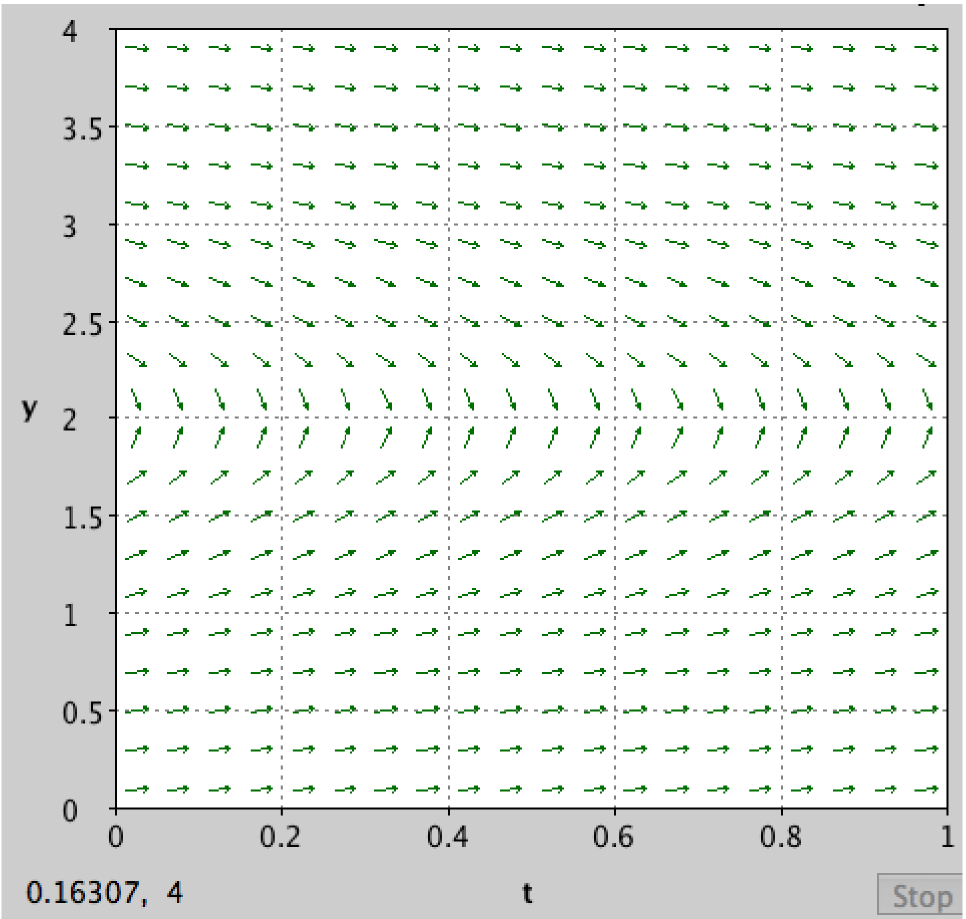
- $f'(y_0) < 0$ then y_0 is a sink
- $f'(y_0) > 0$ then y_0 is a source
- $f'(y_0) = 0$ then we need additional information to determine what kind of equilibrium we have at y_0

Example: Revisit $\frac{dw}{dt} = 3w^3 - 12w^2$

Another example that shows solutions don't always have to exist forever:

$$\frac{dy}{dt} = -\frac{1}{y-2}$$

Note that dy/dt is not defined at $y = 2$. Look at the slope field below:



Start a solution anywhere and it runs into $y = 2$ after which it fails to exist because $y(t)$ has an undefined slope at $y = 2$ (solutions fall into a hole at $y = 2$).

This simply serves as another illustration of how solutions can fail to exist after a period of time. One of the things we'll eventually see is some conditions that guarantee a solution to exist (and a solution to be unique).

Qualitative methods allow us to understand/estimate how solutions behave for autonomous equations and for general equations if we use slope fields, but one of the shortcomings of qualitative methods is that you don't really have any idea of the numerical value of $y(t)$.

1.4 Numerical Techniques - Euler's Method

We want to solve an initial value problem

$$\frac{dy}{dt} = f(t, y), y(t_0) = y_0.$$

A numerical solution starts with the point (t_0, y_0) and consists of a sequence of points $(t_0, y_0), (t_1, y_1), (t_2, y_2), \ldots$ that lie approximately on the true solution $y(t)$ that goes through the point (t_0, y_0).

IDEA: The differential equation gives the slope of the tangent line to the curve. From an approximate point on the solution curve, follow the tangent line to the next approximate point on the curve.

Picture:

Equations:

Euler's Method:
To advance the solution from (t_k, y_k) to (t_{k+1}, y_{k+1}) with step $\Delta t = t_{k+1} - t_k$:

$$y_{k+1} = y_k + \Delta t f(t_k, y_k).$$

Example: Solve $\frac{dy}{dt} = y$ with $y(0) = 1$ for $0 \leq t \leq 1$. Use $\Delta t = 0.1$. Compare to the analytic solution.

1.4: 1,3,6,13,14

s IDEA: give some conditions that guarantee a differntial equation has at least one solution, or has only one solution.

We've already seen situations in which the solution to a D.E. fails to exist after a period of time. Why? What is the issue with the equation that causes these things to happen? Generally, it is because the slope, $y'(t)$, of the solutien, $y(t)$, has a problem in some way.

Example: $\frac{dy}{dt} = -\frac{1}{y-2}$. The D.E. itself shows us that the $y'(t)$ is undefined at any point for which $y = 2$. Thus a solution can never exist when $y = 2$.

Mathematicians develop what are called "existence and uniqueness" theorems:

- these theoroms give conditions that guarantee the existence of solutions to the D.E.
- The are usually *sufficient* conditions. This means conditions that guarantee the existence (or existence and uniqueness), but the conditions are not usually *necessary*, that is, there may be solutions when conditions are not satisfied.

Existence Theorem: (page 66 BDH) - essentially, if $\frac{dy}{dt} = f(t, y)$ and $f(t, y)$ is continous in an (open) rectangle containing (t_0, y_0), then there is a solution $y(t)$ for some short time interval $(t_0 - \epsilon, t_0 + \epsilon)$

IDEA: As long as $f(t, y)$ (the slope function) has no discontinuities in some neighborhood surrounding the point (t_0, y_0), then the solution can be extended through the point (t_0, y_0), at least for a little while

PROBLEM: ϵ may be *very* small, that is the solution may only be extended for a very small time interval. This is a necessary restriction since solutions can blow up, etc.

Stronger conditions are required to guarantee that a unique solution exists:

Uniqueness Theorem: (page 68 BDH) - if f and $\frac{\partial f}{\partial y}$ are both continuous in an open rectangle containing (t_0, y_0), then there is a unique solution through that point (if y_1 and y_2 are two solutions on the short time interval, then $y_1 = y_2$ on that interval).

Water draining from a tank: Suppose we have an empty water tank. The ground is wet near the drain so we know it drained recently. Did it just finish draining, or did finish draining 5 minutes ago? 10 minutes? There are many possible solutions that fit the data and they are all mathematically suitable.

Uniqueness isn't only an issue for textbooks and mathematicans.

Toricelli's Law: The velocity of fluid leaving the tank is related to the height of fluid in the tank by

$$v = \sqrt{2gh},$$

where v is velocity, g is gravity, and h is the height of fluid in the tank. We can use this to derive a differential equation governing the volume of liquid in a tank as it drains.
Assume the tank is a right cylinder with cross-sectional area A, so that the volume of fluid to height h is given by $V = Ah$:

Assume the cross-sectional area of the drain outlet is a.

$$y' = ay + b$$

where a and b can be functions of t, that is, $\frac{dy}{dt} = a(t)y + b(t)$.

The more conventional way to write this equation is

$$\frac{dy}{dt} + p(t)y = q(t)$$

The term on the right hand side, $q(t)$, is called the source term or the forcing term.

If $q(t) = 0$, then we call this a **homogeneous** linear equation and is an easy, separable equation to solve. If we can find one solution, $y(t)$, then all multiples, $ky(t)$ are also solutions (VERIFY) - this is called the **Linearity Principle.**

If $q(t) \neq 0$, then the linear equation is **inhomogeneous** or **nonhomogenous** or sometimes we say the equation is "forced." The Linearity Principle does not apply here, but it can be shown (homework problem) that all solutions have the form $y(t) = ky_h(t) + y_p(t)$, where y_h solves the homogeneous problem, and y_p is one solution to the inhomogeneous problem. Therefore to find the general solution we need to find both y_h and y_p. We already know how to find y_h : set $q = 0$ and solve the separable equation for y_h. We'll focus on solving for y_p :

Finding the particular solution y_p - integrating factor

Consider our differential equation:

$$y' + py = q$$

IDEA: since the left side looks a bit like the product rule, we are going to multiply by a function, $\mu(t)$, called an integrating factor, that is specially chosen so that the left side becomes exactly like the product rule.

$$\mu y' + \mu p y = \mu q$$

Now, if we could arrange it so that $\mu p = \mu'$, then we would get

$$\mu y' + \mu' y = \mu q.$$

Notice the left side is exactly the product rule applied to the product μy:

$$(\mu y)' = \mu q$$

Or

$$\frac{d}{dt}(\mu y) = \mu q.$$

Now integrate both sides:

$$\int \frac{d}{dt}(\mu y)\, dt = \int \mu q\, dt$$

to get

$$\mu y = \int \mu q\, dt$$

Solving for $y(t)$ gives

$$y(t) = \frac{1}{\mu} \int \mu q \, dt$$

If we can do the integral, then great. If not, then we still have at least this form of the solution:

$$y(t) = \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s) q(s) \, ds$$

To find the integrating factor mu we have to solve the separable equation $\mu' = \mu p$:

$$\frac{d\mu}{dt} = \mu p$$

$$\frac{1}{\mu} \, d\mu = p \, dt$$

$$\ln |\mu| = \int p \, dt + C$$

$$\mu = k e^{\int p \, dt}$$

Choose $k = 1$ for simplicity (just need one integrating factor)

$$\mu = e^{\int p \, dt}$$

<u>Example:</u> Find the particular solution for $\frac{dy}{dt} = \frac{3}{t} y + t^5$

Example: Find the particular solution for $\frac{dy}{dt} = t^2 y + 4$

Existence and Uniqueness for Linear D.E.

If p and q are continous on (a, b) and t_0 in (a, b) then there exists a unique solution $y = \phi(t)$ on (a, b) that satisfies

$$\frac{dy}{dt} + p(t)y = q(t),\ y(t_0) = y_0$$

Second Order Linear Equations - Chapter 3 (Boyce and DiPrima)

Second order if

$$y'' = f(t, y, y')$$

Linear if

$$f(t, y, y') = f(t) - p(t)y' - q(t)y$$

so that the whole equation becomes

$$y'' + p(t)y' + q(t)y = g.$$

A related way to write a linear second order equation is

$$P(t)y'' + Q(t)y' + R(t)y = G(t).$$

The equation is homogeneous if $G(t) = 0$ and nonhomogeneous otherwise.

To solve an initial value problem (IVP) for a second order lienar equation we are required to specify two pieces of information. Generally $y(t_0) = y_0$ and $y'(t_0) = y'_0$. (Imagine you have an equation relating acceleration to position and velocity and you want to solve for the position function, then you need to know both the initial position and the initial velocity.)

Initially we'll consider the constant coeffient case homogeneous case

$$ay'' + by' + cy = 0$$

(you considered this case in MTH309 where you solved it by changing it into a 2×2 system.)

For the constant coefficient, homogeneous case, a solution will have the form $y(t) = e^{rt}$. Does this make sense??? More solutions can be made by taking linear combinations of solutions of this form.

<u>Example:</u> (# 2, pg. 142 (Boyce and DiPrima))

$$y'' + 3y' + 2y = 0$$

What is the general solution?

Example: (# 14, pg. 142) $2y'' + y' - 4y = 0$, $y(0) = 0, y'(0) = 1$

Fundamental solutions of second order, linear, homogeneous equations

---

In the last section we considered the constant coefficient case:

$$ay'' + by' + cy = 0.$$

Now we consider the case where coefficients depend on the independent variable, but the equation is homogeneous:

$$y'' + p(t)y' + q(t)y = 0. \tag{1}$$

In this section we'll see when this DE has unique solutions, how to make more solutions by combining exisiting solutions, and when we can solve the corresponding initial value problem.

The existence and uniqueness theorem applies to the homogeneous and nonhomogeneous cases even though throughout the rest of this section we will consider the homogeneous case.

Existence and Uniqueness Theorem: Consider the initial value problem:

$$y'' + p(t)y' + q(t)y = g(t), \; y(t_0) = y_0, \, y'(t_0) = y'_0,$$

where p, q, and g are continuous on open interval I containing t_0. Then there is exactly one solution $y = \phi(t)$ of this problem on the interval I.

Example: Problem 8, page 151.

$$(t-1)y'' - 3ty' + 4y = \sin t, \; y(-2) = 2, \, y'(-2) = 1$$

If we have two solution of (1), y_1 and y_2, then linear combinations of y_1 and y_2 are also solutions. In the language of D.E.s this is called the:

<u>Principle of Superposition:</u> If y_1 and y_2 are solutions of (1),

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constants c_1 and c_2.

<u>Proof:</u>

So, given a pair of solutions y_1 and y_2 to start, then there is a two parameter family of solutions of the form $c_1y_1 + c_2y_2$. Among all these possible solutions, can we find a solution to the IVP:

$$y'' + p(t)y' + q(t)y = 0 \ y(t_0) = y_0, y'(t_0) = y_0'? \tag{2}$$

How do we adjust c_1 and c_2 to match the initial condition?

Theorem 3.2.3: Suppose that y_1 and y_2 are solutions of (1):

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

and that the Wronskian

$$W = y_1y_2' - y_1'y_2$$

is not zero at the point t_0 where the initial conditions (2)

$$y(t_0) = y_0 \qquad y'(t_0) = y_0',$$

are assigned. Then there is a choice of constants c_1, c_2 for which $y = c_1y_1(t) + c_2y_2(t)$ satisfies the differential equation with initial conditions (1).

The amazing thing is that if we have a pair of solutions, y_1, y_2, to (1) and if the Wronskian of this pair of solutions is not zero everywhere on the interval I, then y_1, y_2 form a basis for all possible solutions on I. In D.E. language y_1 and y_2 form a **fundamental set of solutions.**

Theorem 3.2.4: If y_1 and y_2 are two solutions of the differential equation (1):

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

and if there is a point t_0 where the Wronskian of y_1 and y_2 is nonzero, then the family of solutions

$$y = c_1 y_1(t) + c_2 y_2(t)$$

includes every possible solution of (1).

Proof:

Example: Show that $y_1(t) = e^t$ and $y_2(t) = te^t$ are solutions to $y'' - 2y' + y = 0$ and form a fundamental solution set.

Theorem 3.2.5: A good way to construct a fundamental solution set for

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

is to pick a t_0 in I and solve two problems.

To find y_1 solve the IVP:

$$y(t_0) = 1, \qquad y'(t_0) = 0.$$

To find y_2 solve the IVP:

$$y(t_0) = 0, \qquad y'(t_0) = 1.$$

Proof:

Example: Construct a fundamental solution set (as specified by 3.2.5) for

$$y'' + y' - 2y = 0, \quad t_0 = 0.$$

3.3 Linear Independence and the Wronskian

... remember linear algeba ...

Consider a second order, linear, homogeneous differential equation:

y" + p(t) y' + q(t) y = 0, (3)

where $p(t)$ and $q(t)$ are continuous on an open interval I.

The set of all real-valued functions defined on the interval I is a vector space. Of course, not all of those functions will be solutions to (3), but some will. The subset of functions that solve (3) is a two-dimensional subspace of the larger vector space, therefore we need two basis "vectors" or functions to span this solution subspace. In the world of differential equations, these basis functions are called fundamental solutions.

Recall the idea of *linear independence*: two nonzero vectors $\vec{x}_1$ and $\vec{x}_2$ are linearly independent if one is not a multiple of the other. In math terms we say that the vectors are linearly independent if the only solution to the equation:

$$c_1\vec{x}_1 + c_2\vec{x}_2 = \vec{0}$$

is the trivial solution $c_1 = c_2 = 0$. If there is a nontrivial solution, then the vectors are linearly dependent.

The same idea holds for functions: two nonzero functions $f_1(x)$ and $f_2(x)$ are linearly independent on I if the only solution to the equation:

$$c_1 f_1(x) + c_2 f_2(x) = 0$$

is the trivial solution $c_1 = c_2 = 0$. Essentially this means that one function is not a multiple of the other.

To check functions for linear independence we can work directly from the definition (see page 153 for examples) or, for differentiable functions, there is an easier way:

Theorem 3.3.1: If f and g are differentiable functions on an open interval I and if $W(f,g)(t_0) \neq 0$ for some point t_0 in I, then f and g are linearly independent on I. Moreover, if f and g are linearly dependent on I, then $W(f,g)(t) = 0$ for every t in I.

Proof: Dust off your linear algebra skills ...

Example: $f(t) = e^t, g(t) = e^{2t}$

Example: $f(x) = 2x + 1, g(x) = 4x + 2$

Amazingly, we can predict the Wronskian of a pair of solutions y_1 and y_2 to our homogeneous DE, without even actually knowing the solutions. This turns out to be surprisingly useful.

Abel's Theorem (3.3.2): If y_1 and y_2 are solutions to

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where $p(t)$ and $q(t)$ are continous on the open interval I, then the Wronskian $W(y_1, y_2)(t)$ is given by

$$W(y_1, y_2)(t) = c \cdot \exp\left[-\int p(t)\,dt\right],$$

where c is a constant that depends on y_1 and y_2 but not on t. It follows that $W(y_1, y_2)(t)$ is either always zero or always nonzero on I.

Proof:

Example: Find the Wronskian of two solutions without solving the DE:

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0 \qquad \text{Legendre's equation}$$

Example: Abel's Theorem can be used to find a *second* solution to a homogeneous second order equation. For example, consider

$$t^2y'' + 2ty' - 2y = 0, \quad t > 0; \quad y_1(t) = t.$$

Use Abel's theorem to find a second solution.

.

Theorem 3.3.3 - Linear Independence of Solutions: Let y_1 and y_2 be solutions of (3):

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous on an open interval I. Then y_1 and y_2 are linearly independent on I if only if $W(y_1, y_2)(t) \neq 0$ for all t in I. Alternatively, y_1 and y_2 are linearly dependent on I if and only iff $W(y_1, y_2) =$ for all all t in I.

Proof:

Summary - if y_1 and y_2 are solutions of (3) on the open interval I, then the following four things are equivalent (a Differntial Equations Snowball ...)

- y_1 and y_2 are a fundamental solution set on I
- y_1 and y_2 are linearly independent on I
- $W(y_1, y_2)(t_0) \neq 0$ for some t_0 in I
- $W(y_1, y_2)(t) \neq 0$ for all t in I

3.4 2^{nd} order, constant coefficient, homogeneous, **complex** root HW: 7, 11, 17, 19, 25(a)(b), 28

Example: $y'' - 4y' + 13y = 0$

Euler's Formula: $e^{ix} = \cos x + i \sin x, \qquad i = \sqrt{-1}$

Now to write real-value solutions:

3.5 Reduction of Order (repeated roots and more...)

We've seen how to deal with the constant coefficient, homogeneous, second order DE:

$$ay'' + by' + cy = 0$$

in almost every case.

- if $b^2 - 4ac > 0$ then there will be two distinct real roots r_1 and r_2 and a fundamental solution set is $y_1(t) = e^{r_1 t}, y_2(t) = e^{r_2 t}$.
- if $b^2 - 4ac < 0$ then there are a pair of complex conjugate roots $r = \alpha \pm \beta$ and a fundamental solution set is $y_1(t) = e^{\alpha t} \cos \beta t, y_2(t) = e^{\alpha t} \sin \beta t$.
- if $b^2 - 4ac = 0$ this is called the repeated root case because there is one (double) root $r = -\frac{b}{2a}$. As will be shown below, using **reduction of order**, a fundamental solution set is $y_1(t) = e^{rt}, y_2(t) = te^{rt}$.

Example: $y'' + 2y' + y = 0$. Making the usual guess $y = e^{rt}$ leads to the characteristic equation $r^2 + 2r + 1 = (r+1)^2 = 0$. (Note that $b^2 - 4ac = 0$. So one solution will be $y_1(t) = e^{-t}$. But since $r = -1$ is a repeated root, we'll need something different to have a fundamental solution set. A reasonable guess is $y_2(t) = v(t)e^{-t}$. To see if this can be made to work somehow, we'll plug it into the DE and see where it goes.

$$\begin{aligned} y_2(t) &= ve^{-t} \\ y_2'(t) &= v'e^{-t} - ve^{-t} = (v' - v)e^{-t} \\ y_2''(t) &= -(v' - v)e^{-t} + (v'' - v')e^{-t} = (v'' - 2v' + v)e^{-t} \end{aligned}$$

Plugging $y_2(t)$ into the the DE:

$$\begin{aligned} y_2'' + 2y_2' + y_2 &= 0 \\ (v'' - 2v' + v)e^{-t} + 2(v' - v)e^{-t} + ve^{-t} &= 0 \\ (v'' - 2v' + 2v' + v - 2v + v)e^{-t} &= 0 \\ v''e^{-t} &= 0 \\ v'' &= 0 \\ v(t) &= at + b \end{aligned}$$

for any choice of a and b will give a solution to the DE. That is, $y_2(t) = (at + b)e^{-t}$ solves the DE for any a and b. Setting $a = 1$ and $b = 0$ gives a nice, linearly independent choice for $y_2(t) = te^{-t}$. So a fundamental solution set is $y_1(t) = e^{-t}, y_2(t) = e^{-t}$.

Constant coefficient, repeated roots, general case:
$ay'' + by' + cy = 0$ and $b^2 - 4ac = 0$. The (double) root is $r = \frac{-b}{2a}$. The fundamental solution set is:

$$y_1(t) = e^{rt}, \qquad y_2(t) = te^{rt}$$

IVPs with repeated roots are very easy to solve, particularly if the initial data is given at $t = 0$.

Example:

$$y'' + 4y' + 4y = 0, \qquad y(0) = 1, \quad y'(0) = 2.$$

For the general case $y'' + p(t)y' + q(t)y = 0$ if we know one solution $y_1(t)$ we can use Abel's theorem to find the Wronskian and then generate a second solution. Reduction of order gives us a second, different approach, to finding another solution.

Reduction of order - general case: Given the DE $y'' + p(t)y' + q(t)y = 0$ and one solution $y_1(t)$, let $y_2(t) = v(t)y_1(t) = vy_1$.

Example: $t^2y'' + 2ty' - 2y = 0, \quad t > 0, \quad y_1(t) = t.$

HW: 1, 3, 6, 12, 14, 15(a)(b), 18, 23*, 25*, 28*
* try to solve some of these by BOTH Abel's theorem and by Reduction of Order. You should know both approaches.

3.6 Method of undetermined coeffiecients

If the equation isn't homogeneous and is simple enough, then we can guess the form of a solution and adjust it. Here we are considering a nonhomogeneous equation of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

where $p(t), q(t)$ and $g(t)$ are continuous on an open interval I. If $g(t) = 0$ then we call this the homogeneous version of the equation.

Theorem 3.6.2 The general solution of the DE can be written as

$$y = \phi(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

where y_1 and y_2 are a fundamental solution set for the homogeneous version of the equation and Y is one particular solution to the nonhomogoneous version.

Thus to solve the nonhomogeneous DE we need three steps:
1. Find the general solution to the homogeneous DE.
2. Find a particular solution to the nonhomgeneous DE.
3. Add these solutions together.

In the constant coefficient case we can use the *method of undetermined coefficients* to solve Step 2:

$$ay'' + by' + cy = g(t),$$

BUT $g(t)$ must be NICE.

Example: Find a particular solution of $y'' + y' - 2y = 5e^{3t}$

Example: Find a particular solution of $y'' + 5y' + 6y = 2\cos 3t$

Example: Find a particular solution of $y'' + 5y' + 6y = 4te^t$

Example: Find a particular solution of $y'' + 6y' + 9y = 5te^{-3t}$

HW: 1-6, 8, 16, 17

3.7 Variation of Parameters

For the nonhomogeneous second order case, we can only deal with constant coefficients and a NICE right hand side. What if the right hand side (the forcing term) is not nice?

The Variation of Parameters approach is a little like Reduction of Order, except that we start with two solutions to the homoogeneous problem (constant coefficient or not!) y_1 and y_2. Then we make the *smart* guess for the form of the particular solution:

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t).$$

If we do this right we may be able to find the functions u_1 and u_2 to build the particular solution. Since $y(t)$ has to satisfy the nonhomogeneous DE, that will give us ONE equation to satisfy for determining u_1 and u_2, but we'll need a second condition to solve for u_1 and u_2 completely. We can PICK that condition to make the computations simpler.

Example:

$$y'' + y = \sec t$$

First find fundamental solution set for the associated homogeneous problem

Now find a the particular solution to the homogeneous problem by variation of parameters (undetermined coefficients does not work here).

Finally, we can assemble the general solution to the nonhomogeneous problem.

These results can be streamlined in as in the following theorem:

Theorem 3.7.1: If the functions p, q, and g are continuous on an oopen interval I and if the functions y_1 and y_2 are a fundamental solution set of the homogeneous equation associated to the nonhomogeneous equation:

$$y'' + p(t)y' + q(t)y = g(t),$$

then a particular solution of the nonhomogeneous equation is

$$y(t) = -y_1(t) \int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}\, dt + y_2(t) \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}\, dt$$

Example: $y'' + 4y = 3 \csc 2t$

HW: 3.7: 3, 5, 7, 9, 13, 17, 19

$$ay'' + by' + cy = g(t), \qquad y(0) = y_0, y'(0) = y'_0$$

may apply to more than one physical problem.

Spring-mass system (assumes small displacement)

Hooke's Law - the force exerted by the spring is proportional to the displacement:

Dynamic Equations:

- weight:

- spring force:

- damping force modeling friction is proportional velocity and acts opposite motion:

- external forcing such as pushing or an external field that is driving the system:

Example: A mass weighing 8 lb stretches a spring 2 inches. If the mass is pushed upward 2 inches and set in motion with an initial velocity of 1 ft/sec. Determine the position function (there is no damping).

Cases:

$$mu'' + \gamma u + ku = F \to mr^2 + \gamma r + k = 0 \to r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

- **overdamped**

- **underdamped**

- **critically damped**

A similar equation arises in electrical circuits:

$$mu'' + \gamma u + ku = F(t) \qquad LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

Spring system	Electrical circuit
u = displacement	Q = charge
u' = velocity	$Q' = I$ = current
m = mass	L = inductance
γ = damping constant	R = resistance
k = spring constant	$1/C$ = elastance
$F(t)$ = external force	$E(t)$ = electromotive force

More commonly, C is called capacitance. You may see one or two of these in the homework, but unless you're fresh from a physics/circuits course, these may be harder to get your mind around. We'll mostly focus on the spring system.

HW: 1, 5, 7, 9, 10, 11, 19 (more may be added later from 3.9 - Forced Oscillations)

5.1 Power Series Review

For second order DE's our solution techniques are limited to the constant coefficient case, though if we know one solution we can find another. We still don't have a technique to solve something like:

$$(1-x^2)y'' + xy' + y = 0.$$

Power series to the rescue ...

Instead of replicating the review in the book, we'll just refer to it and work problems. You should carefully read points 1.-10. on pages 244-246. We'll look at some.

Example: Where does the following series converge (for what values of x)? $\sum_{n=0}^{\infty} \frac{n}{2^n} x^n$

Example: Find the radius of convergence (and the interval) for $\sum_{n=1}^{\infty} \frac{(x-x_0)^n}{n}$

Example: Multiplying series: $\sin x \cdot \cos x$

Example: Find a Taylor Series expansion: $f(x) = e^x$ at $x_0 = 0$

Example: Shifting indices: $\sum_{n=0}^{\infty} a_n x^{n+2}$

Example: Shifting indices: $(1 - x^2) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Example: Shifting indices: $\sum_{n=1}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n$

5.2 Series Solutions near an Ordinary Point

HW 5.2: 1, 3, 5, 9, 13, 17

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

with P, Q, R continuous and having no common factors. An ordinary point is simply a point x_0 such that $P(x_0) \neq 0$.

Example: Solve $y'' + y = 0$ (we already know the general solution is $y(x) = c_1 \sin x + c_2 \cos x$.)

Power series solution at $x_0 = 0$:

we make the guess $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and then try to determine values for the a_n.

we'll assume that the series converges for some radius of convergence ρ (that is, the series converges absolutely for $|x| < \rho$

Example: $(1 - x^2)y'' - xy' + y = 0; \qquad x_0 = 0$

Example: Solve an IVP.

$$(2+x^2)y'' - xy' + 4y = 0, \qquad y(0) = -1, y'(0) = 3$$

In the previous section we assumed that their was a solution to

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

with radius of convergence $\rho > 0$ as long as $P(x_0) \neq 0$ (x_0 is an ordinary point).

However, how do we *know* such a solution exists? We've already seen that if a function such as $y(x)$ has a power series representation, then the coefficients a_n are related to the derivatives of $y(x)$ by the following formula (see section 5.1):

$$a_n = \frac{y^{(n)}(x_0)}{n!}.$$

So $a_0 = y(x_0)$ and $a_1 = y'(x_0)$ are determined by the initial conditions, but how do we know that we can find the remainder of the a_n for $n \geq 2$ and that the series will converge?

a_2 :

a_3 :

So as long as $p(x)$ and $q(x)$ have infinitely many derivatives at x_0 we can determine all of the a_n. Another way to state this condition is to say that $p(x)$ and $q(x)$ have power series representations (they are analytic) at x_0.

Theorem 5.3.1 If x_0 is an ordinary point of the differential equation:

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

that is, if $p = Q/P$ and $q = R/P$ are analytic at x_0, then the general solution of the DE is

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 y_1(x) + a_1 y_2(x),$$

where a_0 and a_1 are arbitrary, and y_1 and y_2 are linearly independent series solutions that are analytic at x_0. Further, the radius of convergence of each of the series solutions y_1 and y_2 is at least as large as the minimum of the radii of convergence fo the series for p and q.

So to determine the radius of convergence for the series solution to the DE, we need to know the radius of converence for p and q. The easiest way to determine this is to find the distance from x_0 to any singularities (blow ups) in the complex plane for p and q.

Example: Radius of convergence for Taylor series for $\dfrac{1}{1+x^2}$ at $x = 0$?

Example: Radius of convergence for Taylor series of $(x^2 - 8x + 20)^{-1}$ at $x = 0$?

Example: Determine a lower bound on the radius of convergence for series solutions to

$$(x^2 - 2x - 3)y'' + xy' + 4y$$

centered at $x_0 = 4$?

centered at $x_0 = -4$?

centered at $x_0 = 0$?

Example: An alternate approach for determining a few terms in a power series represenation ...

$$(2 + x^2)y'' - xy' + 4y = 0, \qquad y(0) = -1, y'(0) = 3$$

Some Useful Series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{k=0}^{\infty} x^k, \qquad |x| < 1 \qquad \text{Geometric Series}$$

Note: Let $S = 1 + x + x^2 + x^3 + \ldots$
then $S \cdot x = x + x^2 + x^3 + x^4 + \ldots$
So $S - S \cdot x = 1 \to S(1-x) = 1 \to S = \frac{1}{1-x}$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad -\infty < x < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^{2k+1} x^{2k+1}}{(2k+1)!} \qquad -\infty < x\infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^{2k} x^{2k}}{(2k)!} \qquad -\infty < x\infty$$

Note - the last three series are connected through Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots = \sum_{k=0}^{\infty} (-1)^k x^k, \qquad |x| < 1$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k} \qquad -1 < x \leq 1$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ldots = \sum_{k=0}^{\infty} (-1)^k x^{2k} \qquad |x| < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \ldots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k-1}}{2k-1} \qquad |x| < 1$$

For the homogeneous second order DE

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

we know we can solve the DE, perhaps finding a power series solution at x_0, if $P(x_0) \neq 0$ (x_0 is an ordinary point).

What if $P(x_0) = 0$? If $P(x_0) = 0$ we call x_0 a singular point.

Can we solve the DE in the neigborhood of a singular point? (Find a series type solution centered at x_0.) The answer may be yes depending on the nature of the singular point.

Definition - Regular Singular Point: If $P(x_0) = 0$ but both limits:

$$\lim_{x \to x_0} (x - x_0)\frac{Q(x)}{P(x)}, \qquad \lim_{x \to x_0} (x - x_0)^2\frac{R(x)}{P(x)}$$

exist and are finite, then x_0 is a *regular singular point.* If either limit does not exist or is not finite, then x_0 is an *irregular singular point.*

Example: $x^2(1 - x^2)y'' + 2xy' + 4y = 0$
Investigate and classify the singular points.

$$L[y] = x^2 y'' + \alpha x y' + \beta y = 0$$

$x = 0$ is a regular singular point and this equation serves as a model or template for all regular singular points. We'll assume $x > 0$ and then extend to $x < 0$ at the end.

Once again, we guess: $y = x^r$

$$\begin{aligned} L[x^r] &= x^2(x^r)'' + \alpha x(x^r)' + \beta(x^r) \\ &= x^2 r(r-1)x^{r-2} + \alpha x r x^{r-1} + \beta x^r \\ &= r(r-1)x^r + \alpha r x^r + \beta x^r \\ &= (r(r-1) + \alpha r + \beta)x^r = 0 \end{aligned}$$

For x^r to be a solution we need to solve the *indicial equation*:

$$F(r) = r(r-1) + \alpha r + \beta = 0$$

This is a quadratic equation in the variable r. Just as in constant coefficient second order DE scenario, there are three cases - two distinct real roots, repeated real roots, complex roots.

Case 1 real distinct roots r_1 and r_2 so general solution is

$$y = c_1 x^{r_1} + c_2 x^{r_2}$$

Case 2 repeated real root $r = r_1 = r_2$

$$y = c_1 x^r + c_2 (\ln x) x^r$$

Case 3 $r_{1,2} = \lambda \pm \mu i$

$$x^{r_1} = x^{\lambda + \mu i} = e^{\ln x^{\lambda + \mu i}} = e^{(\lambda + \mu i)\ln x} = e^{\lambda \ln x} e^{i\mu \ln x} = x^\lambda(\cos(\mu \ln x) + i \sin(\mu \ln x)) \rightarrow$$

$$y = x^\lambda (c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x)$$

Extending to $x < 0$: can work out any of these solution forms on any interval not containing 0 by simply replacing x by $|x|$

Example 1: $2x^2y'' + 3xy' - 15y = 0, \quad y(1) = 0, \quad y'(1) = 1$

Example 2: $x^2y'' - 7xy' + 16y = 0$ find the general solution.

Example 3: $x^2y'' + 3xy' + 4y = 0$ find the general solution

Improper Integrals: Recall:

$$\int_a^\infty f(t)\,dt = \lim_{A\to\infty}\int_a^A f(t)\,dt$$

If this limit exists we say the integral converges, otherwise we say it diverges.

Example: $\displaystyle\int_1^\infty t^{-p}\,dt$

Example: $\displaystyle\int_0^\infty e^{ct}\,dt$

Comparison Theorem for Improper Integrals (Theorem: 6.1.1):
If f is piecewise continuous for $t \geq a$ and if $|f(t)| \leq g(t)$ for $t \geq M$ for some positive M and $\int_M^\infty g(t)\,dt$ converges, then $\int_a^\infty f(t), dt$ also converges

On the other hand, if $f(t) \geq g(t) \geq 0$ for $t \geq M$ and $\int_M^\infty g(t)\,dt$ then $\int_a^\infty f(t)\,dt$ diverges

---

The Laplace transform is one of many kinds of integral transforms used in applied math, physics, engineering, etc.

$$\mathcal{L}\left\{f(t)\right\}(s) = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

This is one example of a more general sort of transform wherein one starts with a function of t say $f(t)$ and transforms it into a function of s:

$$F(s) = \int_\alpha^\beta K(s,t) f(t)\,dt$$

Another very important example of this sort of transform is the Fourier transform:

$$F(\omega) = \int_{-\infty}^\infty e^{i\omega t} f(t)\,dt$$

For the Laplace transform it is possible to allow complex values of s (for which the Laplace transform and the Fourier transform are closely related), but we don't do that in this book.

The Laplace transform of a function exists as long as the function grows no faster than an exponential:

Theorem: 6.1.2: If

1. f is piecewise continuous on $[0, A]$ for any $A > 0$
2. the tail of f is bounded by an exponential, that is, there is a $K > 0, M > 0$ and a so that $|f(t)| \leq Ke^{at}$ when $t \geq M$, then the Laplace transform exists for $s \geq a$.

Proof: This is really about being able to evaluate the integrals, some of which are improper. For the improper integrals we have to know that they will converge:

Example: $\mathcal{L}\{1\}(s) =$

Example: $\mathcal{L}\{e^{at}\}(s)$

Example: $\mathcal{L}\{\cos at\}(s)$

HW 6.1: 1, 2, 5, 7, 10, 12, 13, 15, 16, 21-24

6.2 Initial Value Problems and Laplace Transforms

HW 6.2: 1, 3, 5, 7, 8, 11, 13, 17, 18, 21, 24

What do Laplace transforms have to do with differntial equations?

IDEA: Laplace transforming a DE turns it into an algebra problem. We solve the algebra problem and then undo the transformation.

We start by understanding what the Laplace transform does to derivatives:

Theorem 6.2.1: If f is continuous and f' is piecewise on every finite interval $[0, A]$ and f if of exponential order (there are K, a, M so that $|f(t)| \leq Ke^{at}$ for $t \geq M$) then $\mathcal{L}\{f'(t)\}$ exists and (for $s > a$)

$$\boxed{\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0)}$$

Proof:

Corollary 6.2.2: $f, f', \ldots, f^{(n-1)}$ continuous and $f^{(n)}$ piecewise continous on all $[0, A]$ and $f, f', \ldots, f^{(n-1)}$ of exponential order then

$$\mathcal{L}\left\{f^{(n)}(t)\right\}(s) = s^n\mathcal{L}\{f(t)\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-1)}(0) - f^{(n)}(0)$$

e.g. $\mathcal{L}\{f''(t)\}(s) = s^2\mathcal{L}\{f(t)\}(s) - sf(0) - f'(0)$
$\mathcal{L}\left\{f^{(3)}(t)\right\}(s) = s^3\mathcal{L}\{f(t)\}(s) - s^2f(0) - sf'(0) - f''(0)$

Why? We'll see in a minute, but the basic idea is the that the Laplace tranform turns a problem with derivatives into an algebra problem. Solve the algebra problem and then inverse transform.

Example: Solve $y'' + 3y' + 2y = 0, \qquad y(0) = 1, y'(0) = 0$ using Laplace transforms. (Uh oh ... here come some partial fraction decompositions ...)

Example: Solve $y'' + 2y' + 5y = 0, \qquad y(0) = 2, y'(0) = -1$

HW 6.3: 1, 3, 7, 8, 11, 14, 15, 17, 24

In the time domain, we might think of 0 as the state of being “off” and 1 as the state of being “on,” then if we flip a switch to turn the state from “off” to “on” at time $t = c$ we have the unit step function:

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

If we’d rather go from “on” to “off” at time $t = c$?:

How about if we want to be “off” for all t except between 1 and 2?

Laplace transform of unit step function:

$\mathcal{L}\{u_c(t)\} =$

We’ll use these unit step functions to translate functions/signals in the time domain so that they start at different times:

Laplace transform of translated function:

$$g(t) = f(t-c)u_c(t) = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c \end{cases}$$

$\mathcal{L}\{g(t)\} =$

$$\boxed{\mathcal{L}\{u_c(t)f(t-c)\} = e^{-sc}\mathcal{L}\{f(t)\} = e^{-sc}F(s)}$$

$$\boxed{\mathcal{L}^{-1}\{e^{-sc}F(s)\} = u_c(t)f(t-c)}$$

Example: Find the Laplace tranform of

$$g(t) = \begin{cases} 0, & t < 1 \\ t^2 - 2t + 2, & t \geq 1 \end{cases}$$

Example: Find the inverse Laplace transform of

$$F(s) = \frac{2e^{-2s}}{s^2 - 4}$$

Example: Find the Laplace transform of

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \\ 1, & 2 \le t < 3 \\ 0, & 3 \le t \end{cases}$$

HW 1, 3, 5, 10, 12

Example:

$$y'' + 2y' + 2y = h(t), \quad y(0) = 0, y'(0) = 1$$

$$h(t) = \begin{cases} 1, & \pi \le t < 2\pi \\ 0, & 0 \le t < \pi \text{ or } t \ge 2\pi \end{cases}$$

HW 1, 5, 6, 7, 10

Impulse functions are used to model very short bursts like hitting a bell or a short zap to a circuit.

Unit impulse for time 2τ:

$$d_\tau(t) = \begin{cases} \frac{1}{2\tau}, & -\tau < t < \tau \\ 0, & |t| \geq \tau \end{cases} \quad =$$

Total impulse:

$$I(\tau) = \int_{-\infty}^{\infty} d_\tau(t)\, dt = \underline{\hspace{3cm}}$$

Ideal unit impulse "function": Let $\tau \to 0$

We want $\lim_{\tau \to 0} I(\tau) = 1$ and $\lim_{\tau \to 0} d_\tau(t) = 0$ if $t \neq 0$.

In words, we want the "top-hat" function to get infinitely narrow and still have area 1 beneath it. It can no longer truly be a function since no function can be defined at a point and have area 1 beneath it, nevertheless we define the (Dirac) delta function $\delta(t)$ to have impulse of 1 at $t = 0$ but $\delta(t) = 0$ if $t \neq 0$.

$$\boxed{\delta(t) = 0 \text{ if } t \neq 0 \text{ and } \int_{-\infty}^{\infty} \delta(t)\, dt = 1}$$

If we want to move the unit impulse to another point simply translate it as $\delta(t - t_0)$.

Since the Dirac delta function will be used to model instantaneous impulses (forcing) in our DE's we need it's Laplace transform. To get its Laplace transform we'll "prove" a broader result first:

$$\boxed{\int_{-\infty}^{\infty} f(t)\delta(t - t_0)\, dt = f(t_0)}$$

We can use the previous result to establish the Laplace tranform of the Dirac delta function:

Now an example of a DE with an instantaneous impulse:

Example:

$$y'' - y = -20\delta(t-3), \quad y(0) = 1, y'(0) = 0$$

Consider a second order differential equation in which, instead of supplying two pieces of initial data at a particular point, we specify data at two different points. In this setting we often refer to the independent variable as x because we think of it as a spatial variable. The locations, $x = a$ and $x = b$, where the data (function or derivative values) are specified are called the boundaries. This kind of problem is called a **boundary value problem.**

Example: $y'' + 2y = x, \qquad y(0) = 0, y(\pi) = 0$

- Solve the homogeneous problem $y'' + 2y = 0$

  Side note: if we were trying to solve the homogeneous boundary value problem $y'' + 2y = 0, \quad y(0) = y(\pi) = 0$, what would the solution be?

- Find the particular solution of $y'' + 2y = x$

- General solution:

- Find c_1 and c_2 to satisify boundary values (if possible):

Along the way, we saw that the homogeneous problem

$$y'' + 2y = 0, \quad y(0) = y(\pi) = 0$$

has only the trivial solution. How about

$$y'' + y = 0, \quad y(0) = y(\pi) = 0?$$

So the homogeneous problem

$$y'' + \lambda y = 0, \quad y(0) = y(\pi) = 0$$

has nonzero solutions for some values of λ, but not for others. Does this sound familiar?

Remember this problem ... find values λ and nonzero vectors $\vec{x}$ to solve

$$A\vec{x} = \lambda\vec{x}$$

Finding values of λ for which the boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = y(\pi) = 0$$

has nontrivial (nonzero) solutions is an eigenvalue problem. The values of λ that lead to nonzero solutions are called eigenvalues and the corresponding solutions are called **eigenfunctions.** Solve the eigenvalue problem: Find the eigenvalues and eigenfunctions of

$$y'' + \lambda y = 0, \quad y(0) = y(\pi) = 0$$

- $\lambda > 0$

- $\lambda < 0$

- $\lambda = 0$

What if we changed the boundary conditions to $y(0) = y(L) = 0$?

What if we changed the boundary conditions to $y(0) = 0, y'(\pi) = 0$?

Due: Monday, December 5, 2011

And NUH is the letter I use to spell Nutches,
Who live in small caves, known as Nitches, for hutches.
These Nutches have troubles, the biggest of which is
The fact there are many more Nutches than Nitches.
Each Nutch in a Nitch knows that some other Nutch
Would like to move into his Nitch very much.
So each Nutch in a Nitch has to watch that small Nitch
Or Nutches who haven't got Nitches will snitch.

-Dr. Seuss, *On Beyond Zebra* (1955)

In this project, you will study the one-parameter family of nonlinear, first order system consisting of predator-prey equations. The family is

$$\begin{aligned}\frac{dx}{dt} &= 9x - \alpha x^2 - 3xy \\ \frac{dy}{dt} &= -2y + xy\end{aligned}$$

where $\alpha \geq 0$ is a parameter. In other words, for different values of α we have different systems. The variable x is the population (in some scaled units) of prey, and y is the population of predators. For a given value of α, we want to understand what happens to these populations as $t \to \infty$.
You should investigate the phase portraits of these systems for various values of α in the interval $0 \leq \alpha \leq 5$. To get started, you might want to try $\alpha = 0, 1, 2, 3, 4,$ and 5. (Use the phase plane program linked from our HW page.) Think about what the phase portrait means in terms of the evolution of the x and y populations. Where are the equilibrium solutions? What does the Jacobian tell you about their types? What happens to a typical solution curve? Also, consider the behavior of the special solutions where either $x = 0$ or $y = 0$ (solution curves lying on the x- or y-axes).
Determine the bifurcation values of α–that is, the values of α where nearby α's lead to "different" behaviors in the phase portrait. For example $\alpha = 0$ is a bifurcation value because for $\alpha = 0$, the long-tem behavior of the populations is dramatically different than the long-term behavior of the populations if α is slightly positive. The process of finding the equilibrium solutions and classifying for the equation above should suggest bifurcation values. Find all of them.

Your report: After you have determined all of the bifurcation values for α in the interval $0 \leq \alpha \leq 5$, study enough specific values of α to be able to discuss all of the various population evolution scenarios for these systems. In your report, you should describe these scenarios using the phase portraits. Your report should include:

1. A brief discussion of the significance of the various terms in the system. For example, what does $9x$ represent? Why is it positive? What does the $3xy$ term represent? etc.
2. A discussion of all bifurcations including the bifurcation at $\alpha = 0$. For example, a bifurcation occurs between $\alpha = 3$ and $\alpha = 5$. What does this bifurcation mean for the predator population?

Address the following questions in the form of a short essay (I **do not** want you to say the answer question 1 is . . .), and support your assertions with selected illustrations. (Please remember that although one good illustration may be worth 1000 words, 1000 illustrations are usually worth nothing.)

Have questions? Feel free to stop by my office (1026 Cowley Hall) or contact me by email.
Hints

1. Initially, the phase portraits at alpha = 0, 1, 2, 3, 4, 5 are only used to help you see that the dynamics are changing as α changes. Those are not the bifurcation points. You must do some work by hand to find the exact bifurcation points (there are 3 of them in total).

2. Both populations must be non-negative. This condition will restrict the number of the euilibrium solutions leading to two of the bifurcation points.

3. Find the equlibrium solutions as a function of α.

4. Break your analysis up into cases depending on which side of the bifurcation point(s) α lies on.

5. For each case, determine the Jacobian at each equilibrium (the Jacobian will be dependent on α as well) and analyze the type of equilibrium. You do not need to calculate any eigenvectors.

6. Provide a phase portrait (for example the appropriate one from comment 1. above) and description (using the predator-prey language) for each case.

HW 10.2: 4, 9, 13, 15, 22ab

Where a Taylor series was used to approximate a function by something that was "polynomial like," Fourier series are used to approximate periodic functions by decomposing them into a sum of sines and cosines at different frequencies. The basic form of a Fourier series on the interval $[0, 2L]$ or $[-L, L]$ is

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right)$$

This Fourier series will be convergent for some set of x in $[-L, L]$ to a periodic function $f(x)$.

Periodic Functions: the **fundamental period** of a function is the smallest positive value of T for which $f(x + T) = f(x)$.

What is the fundamental period of $\cos(x)$? $\sin(x)$?

$\cos(m\pi x/L)$? $\sin(m\pi x/L)$?

If we have a periodic function $f(x)$, how do we approximate it by a Fourier series? How do we find the coefficients a_m and b_m?

We will proceed by recalling how a similar situation worked with vectors in $\mathbb{R}^n$. Let $\{\vec{x}_1, \vec{x}_2, \ldots \vec{x}_n\}$ be an *orthogonal* basis for $\mathbb{R}^n$.

If $\vec{x}$ is an arbitrary vector in $\mathbb{R}^n$, how do we go about writing it or expanding it in terms of our orthogonal basis for $\mathbb{R}^n$.

How can we mimic this process for a periodic function?

Inner (dot) product for functions:

$$(u, v) = \int_{\alpha}^{\beta} u(x)v(x)\,dx$$

and since we are we are working with periodic functions on the interval $[-L, L]$ we'll use

$$(u, v) = \int_{-L}^{L} u(x)v(x)\,dx$$

Orthogonality relations for sine and cosine:

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases} = L \cdot \delta_{m-n}$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases} = L \cdot \delta_{m-n}$$

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = 0$$

Suppose $f(x)$ is periodic on $[-L, L]$ and assume $f(x)$ has a Fourier series expansion:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right)$$

How do we find the a_m and b_m? Use the dot product or inner product and take advantage of orthogonality, just as we did in $\mathbb{R}^n$.

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx \qquad n \geq 1$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx \qquad n \geq 1$$

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx$$

Example: $f(x) = \begin{cases} x+1, & -1 \le x < 0 \\ 1-x, & 0 \le x < 1 \end{cases}$ $\quad f(x+2) = f(x)$. Find the Fourier series expansion.

Theorem 10.3.1: Suppose that f and f' are piecewise continuous on the interval $-L \leq x < L$. Further, suppose that f is defined outside the interval $-L \leq x < L$ so that it is periodic with period $2L$. Then f has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L} \right),$$

whose coefficients are determined by the integrals in the previous section. The Fourier series converges to $f(x)$ at all points where f is continous, and to $[f(x+) + f(x-)]/2$ at all points where f is discontinous.

Convergence at a jump discontinuity:

This theorem is sufficient (sufficient means that if f satisfies the hypotheses, then f has a convergent Fourier series), but not necessary (if f has a convergent Fourier series, is does not have to satisfy the hypotheses of theorem). Note, there are more general conditions which gaurantee that f has a convergent Fourier series, that is, there are more general sufficient conditions - these theorems are beyond the scope of this class.

Important: Functions with infinite discontinuities or infinitely many finite jumps are excluded from this theorem.

How is this possible? This result is actually difficult for me to understand and surprising. A(n) (infinite) sum of infinitely differentiable functions (really, really smooth) can converge to something that is discontinuous!

10.3 HW: 7, 11

Even Functions If $f(x)$ is even ($f(-x) = f(x)$) and periodic, then it has a cosine series.

Odd Functions: If $f(x)$ is odd ($f(-x) = -f(x)$) and periodic, then it has a sine series.

If we have a function that is not periodic on a finite interval, say $[0, L]$, we can extend it as an even or an odd function to $[-L, L]$ and then repeat it periodically ... now it has a sine or cosine series that will converge to the original function (if it was nice enough) on $[0, L]$.

Example: $f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \end{cases}$ Extend f and find a sine series with period 4.

10.4 HW 1-6, 7, 9, 13, 15, 21, 22

Consider a slender rod which has been initially heated. The ends of the rod are held at a fixed temperature. How does the temperature in the rod behave as a function of time and location with in the rod?

Let the rod be positioned on the interval $[0, L]$ and let $u(x, t)$ represent the temperature of the rod at position x in $[0, L]$ at time t. The equation governing the distribution of heat is

$$u_t = \alpha^2 u_{xx}$$

or

$$\frac{\partial}{\partial t}u(x,t) = \alpha^2 \frac{\partial^2}{dx^2}u(x,t)$$

To completely specify the temperature in the rod, we must know the iniitial temperature distribution (at $t = 0$).

$$u(x, 0) = f(x)$$

and also must specify boundary conditions at the ends of the rod such as

$$u(0, t) = u(L, t) = 0, \quad t > 0$$

(Other boundary conditions are possible, this is just saying to hold the ends of the rod at zero temperature. For a reasonable derivation of the heat equation look it up on Wikipedia.)

Finding nonzero solutions. We'll first try to find a "separable" solution

$$u(x, t) = X(x)T(t).$$

This technique is called separation of variables. We now have two differential equations to solve as candidates for possible solutions.

$$T'(t) = -\lambda T(t), \qquad X''(x) = -\lambda X(x).$$

Notice that we'll need $X(0) = X(L) = 0$ for $u(x,t)$ to satisfy the boundary conditions. Fortunately we have experience with both of these equations.

A sum of solutions satisfying $u_t = \alpha^2 u_{xx}$ and the boundary conditions should also satisfy the partial differential equation and boundary conditions, but how do we satisfy the additional requirement that $u(x,0) = f(x)$?

Example: Solve $u_t = u_{xx}$ with $u(0,t) = u(2,t) = 0, t > 0$ and $u(x,0) = f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 1 < x \leq 2 \end{cases}$.

Suppose we are studying the nonlinear system

$$\frac{dx}{dt} = f(x,y) \qquad \frac{dy}{dt} = g(x,y). \tag{4}$$

We seek a pair of functions $x = x(t)$ and $y = y(t)$ that simultaneously satisfy both equations. We can think of these equations as governing the motion of a particle in the xy-plane as a function of time t.

A special kind of solution is called an **equilibrium solution**. These are solutions which remain constant for all time so that solution stays at the same **equilibrium point** in the xy-plane for all time. To find such a solution we require that $\frac{dx}{dt} = \frac{dy}{dt} = 0$ (no change in x or y). Thus we have to simultaneously solve the equations

$$f(x,y) = 0, \qquad g(x,y) = 0. \tag{5}$$

A solution (x_0, y_0) to (5) is called an equilibrium solution or an equilibrium point.

Example: Find equilibrium solutions for the mutual competition model of rabbits versus sheep (both are competing for the same grass, R and S are measured in the hundreds):

$$\begin{aligned}\frac{dR}{dt} &= 3R - R^2 - 2RS\\ \frac{dS}{dt} &= 2S - S^2 - RS\end{aligned}$$

For nonlinear equations it can be difficult to predict the behavior of solutions (analytically) except in special cases, however for solutions that close to equilibrium points we can make some progress. Suppose $(x(t), y(t))$ is near an equilibrium solution (x_0, y_0). Define the difference between the equilibrium solution and the nearby solution as

$$\Delta x(t) = x(t) - x_0, \quad \Delta y(t) = y(t) - y_0$$

so that

$$x = x_0 + \Delta x, \quad y = y_0 + \Delta y. \tag{6}$$

We've left off the (t) symbols for simplicity. Notice that Δx and Δy depend on t and represent the displacement from the equilibrium solution which constant and independent of t.

Now we substitute (6) into (4) to get

$$\begin{aligned} \frac{d(x_0+\Delta x)}{dt} &= f(x_0+\Delta x, y_0+\Delta y) && (7) \\ \frac{d(y_0+\Delta y)}{dt} &= g(x_0+\Delta x, y_0+\Delta y) && (8) \\ & && (9) \end{aligned}$$

On the left hand side of these equations we use the fact that $\frac{dx_0}{dt} = \frac{dy_0}{dt} = 0$ to write

$$\frac{d(x_0+\Delta x)}{dt} = \frac{dx_0}{dt} + \frac{d\Delta x}{dt} = \frac{d\Delta x}{dt} \tag{10}$$

and

$$\frac{d(y_0+\Delta y)}{dt} = \frac{dy_0}{dt} + \frac{d\Delta y}{dt} = \frac{d\Delta y}{dt} \tag{11}$$

On the right hand side we'll use Taylor Series expansion about the point (x_0, y_0) to write

$$f(x_0+\Delta x, y_0+\Delta y) = f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \text{ higher order terms} \tag{12}$$

The expansion for g is similar. The higher order terms involve higher derivatives and higher powers of Δx and Δy. Notice that if $(x(t), y(t))$ is close to an equilibrium solution (x_0, y_0) then $(\Delta x, \Delta y)$ will both be small and the higher order terms become neglible. Also, $f(x_0, y_0) = g(x_0, y_0) = 0$. So after expanding the right hand side and dropping the higher order terms we have the following system of differential equations governing the evolution of the displacements Δx and Δy from the equilibrium.

$$\begin{aligned} \frac{d\Delta x}{dt} &= f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y && (13) \\ \frac{d\Delta y}{dt} &= g_x(x_0, y_0)\Delta x + g_y(x_0, y_0)\Delta y && (14) \\ & && (15) \end{aligned}$$

The partial derivatives are all evaluated at the equilibrium solution so each of those terms is just a number multiplying Δx or Δy, thus this is a linear system of ODE's:

$$\frac{d}{dt}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \tag{16}$$

The matrix in (16) is called the Jacobian matrix:

$$J(x, y) = \begin{bmatrix} f_x(x, y) & f_y(x, y) \\ g_x(x, y) & g_y(x, y) \end{bmatrix}$$

Example: Find the Jacobian matrix and evaluate it at each of the equilibrium solutions.

$$\begin{aligned} \frac{dR}{dt} &= 3R - R^2 - 2RS \\ \frac{dS}{dt} &= 2S - S^2 - RS \end{aligned}$$

Near the equilibrium at $(3, 0)$ we have

$$\frac{d}{dt}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \tag{17}$$

The general behavior of the (small) displacements Δx and Δy is determined by the eigenvalues of Jacobian matrix, in this case $\lambda_1 = -3, \lambda_2 = -1$, so both eigenvalues are negative. Solutions to this linear system will have the form

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = c_1 e^{-3t}\vec{v}_1 + c_2 e^{-t}\vec{v}_2 \tag{18}$$

where $\vec{v}_1$ and $\vec{v}_2$ are the corresponding eigenvectors of the Jacobian matrix. For our purposes we just want to predict the general behavior or **stability** of the solutions near the equilibrium so it is enough to know that the eigenvalues are negative and that the displacements will decay to zero. So solutions that start near this equilibrium will decay toward this equilibrium - it is called a sink or a stable node.

For a review of the general behavior of linear systems, eigenvalues, and the phase planes, see Section 9.1 in your textbook. Table 9.1.1 on page 492 may be particularly useful.

Another way to characterize these different sorts of solutions for 2D linear systems is to use the Trace-Determinant Plane: