## Homework 3

Solutions

Please show all your work. When possible, write your answers in complete sentences. The easier your solution is to read, the easier it is to give you feedback and points.

1. If $a, b$, and $c$ are positive constants, show that all solutions of $ay'' + by' + cy = 0$ approach zero as $t \to \infty$.

$ar^2 + br + c = 0 \Rightarrow r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If $b^2 - 4ac < 0$ then the roots are complex w/ real part $\lambda = \frac{-b}{2a}$ & imaginary part $\mu = \sqrt{4ac - b}$

the solutions have the form

$y = e^{\lambda t}(c_1 \sin \mu t + c_2 \cos \mu t)$ w/ $\lambda < 0$, so $y(t) \to 0$ as $t \to \infty$. ($\lambda < 0$ because $\lambda = -\frac{b}{2a}$ & $a > 0, b > 0$)

If $b^2 - 4ac > 0$, & $0 < b^2 - 4ac < b^2$ since $a > 0$ & $c > 0$

So $0 < \sqrt{b^2 - 4ac} < b$,

Thus $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} < \dfrac{-b + b}{2a} < 0$

So the largest root of the characteristic equation negative.

This shows both roots, $r_1$ & $r_2$ of the characteristic eq'n are negative.

It follows easily, that solutions $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \to 0$ as $t \to \infty$.

If $b^2 - 4ac = 0$, then $r = -\frac{b}{2a}$ &

$y(t) = c_1 e^{-b/2a\, t} + c_2 t e^{-b/2a\, t} \to 0$ as $t \to \infty$.

2. The solution to the IVP

$$9y'' + 6y' + y = 0, \qquad y(0) = 1, y'(0) = -4$$

becomes negative and dips below -3 before ultimately decaying exponentially toward zero. Find the exact coordinates $(t_0, y_0)$ of the minimum point.

$$r = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 9 \cdot 1}}{2 \cdot 9} = \frac{-6}{18} = \frac{-1}{3}$$

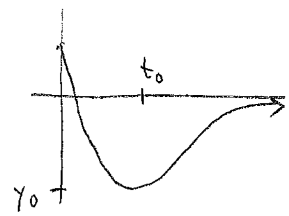
$$y(t) = c_1 e^{-t/3} + c_2 t e^{-t/3}$$

$$y(0) = c_1 + 0c_2 = 1 \Rightarrow c_1 = 1$$

$$y'(0) = -\frac{1}{3}c_1 + c_2\left(-\frac{1}{3}(0) + 1\right) = -\frac{1}{3}c_1 + c_2 = -4$$

$$c_2 = -4 + \frac{1}{3}c_1 = -4 + \frac{1}{3} = \frac{-11}{3}$$

$$y(t) = e^{-t/3} - \frac{11}{3} t e^{-t/3} = \left(1 - \frac{11}{3}t\right)e^{-t/3}$$




min occurs when $y'(t_0) = 0$

$$y'(t) = \frac{-1}{3}\left(1 - \frac{11}{3}t\right)e^{-t/3} - \frac{11}{3}e^{-t/3}$$

$$= \left[-\frac{1}{3}\left(1 - \frac{11}{3}t\right) - \frac{11}{3}\right]e^{-t/3}$$

$$= \left[\frac{-1}{3} + \frac{11}{9}t - \frac{11}{3}\right]e^{-t/3} = \left[\frac{11}{9}t - 4\right]e^{-t/4} = 0$$

if $\frac{11}{9}t = 4 \Rightarrow t = \frac{9}{11} \cdot 4 = \frac{36}{11} \approx 3.2727$

$y'$: $-$ on the left, $0$ at $t = 36/11$, $+$ on the right

min. value is

$$y\left(\frac{36}{11}\right) = \left(1 - \frac{11}{3} \cdot \frac{36}{11}\right)e^{-\frac{36}{11} \cdot \frac{1}{3}} = (1 - 12)e^{-12/11} = -11e^{-12/11} \approx -3.6950$$

$$(t_0, y_0) = \left(\frac{36}{11}, -11e^{-12/11}\right) \approx (3.2727, -3.6950)$$

3. Find a second solution to $t^2y'' + 3ty' + y = 0, \quad t > 0, \quad y_1(t) = t^{-1}$ by two different methods:

(a) Reduction of order:

$$y_2(t) = v(t)\,y_1(t) = vt^{-1}$$

$$y_2' = v(-1)t^{-2} + v't^{-1}, \quad y_2'' = v\,2t^{-3} + v'(-1)t^{-2} + v'(-1)t^{-2} + v''t^{-1}$$

so $y_2 = \frac{v}{t}, \quad y_2' = -\frac{v}{t^2} + \frac{v'}{t}, \quad y_2'' = \frac{2v}{t^3} - \frac{2v'}{t^2} + \frac{v''}{t}$

Plug in:

$$t^2\left(\frac{2v}{t^3} - \frac{2v'}{t^2} + \frac{v''}{t}\right) + 3t\left(\frac{-v}{t^2} + \frac{v'}{t}\right) + \left(\frac{v}{t}\right) = 0$$

$$\frac{2v}{t} - 2v' + v''t - \frac{3v}{t} + 3v' + \frac{v}{t} = 0$$

$$v''t + v' = 0$$

$$t\frac{d}{dt}(v') = -v'$$

$$\frac{1}{v'}\,d(v') = -\frac{1}{t}\,dt$$

$$\ln|v'| = -\ln|t| + C \qquad (t>0)$$

$$v' = c_1 \cdot \frac{1}{t}$$

$$v(t) = c_1 \ln t + c_2$$

so $y_2(t) = \frac{v(t)}{t} = \frac{c_1 \ln t + c_2}{t}$ ← this is the general solution, it includes $y_1(t)$.

~~The~~ Fundamental solutions are

$$\left\{ \frac{\ln t}{t}, \frac{1}{t} \right\}$$

(b) Using the Wronskian found by Abel's Theorem: $y'' + \frac{3}{t}y' + \frac{1}{t^2}y = 0,\ t>0$

$$W = ce^{-\int p(t)\,dt}$$
$$= ce^{-\int \frac{3}{t}\,dt} = ce^{-3\ln t} = ce^{\ln t^{-3}} = ct^{-3} = \frac{c}{t^3} \qquad \star$$

also

$$W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$
$$= \frac{1}{t}y_2' - y_2\left(\frac{-1}{t^2}\right) \qquad \star\star$$

Equating $\star$ & $\star\star$

$$\frac{1}{t}y_2' + \frac{1}{t^2}y_2 = \frac{c}{t^3}$$
$$y_2' + \frac{1}{t}y_2 = \frac{c}{t^2}$$
$$\mu = e^{\int \frac{1}{t}dt} = e^{\ln t} = t$$
$$ty_2' + t\cdot\frac{1}{t}y_2 = t\cdot\frac{c}{t^2}$$
$$ty_2' + y_2 = \frac{c}{t}$$
$$\frac{d}{dt}(ty_2) = \frac{c}{t}$$
$$ty_2 = c\ln t + c_2$$
$$y_2 = \frac{c\ln t}{t} + \frac{c_2}{t}$$

$$y_2 = \frac{\ln t}{t} \qquad (c=1,\ c_2=0)$$

gives linearly indep. $y_2$

4. Solve $y'' - 5y' + 4y = 8e^x + 10x \sin x$.

(a) homog. prob. $r^2 - 5r + 4 = 0$, $(r-4)(r-1) = 0$, $y(t) = c_1 e^x + c_2 e^{4x}$

(b) inhomog. prob 1. $y(x) = Axe^x$, $y' = A(x+1)e^x$, $y'' = A(x+2)e^x$

$$A(x+2)e^x - 5A(x+1)e^x + 4Axe^x = 8e^x$$

$$\cancel{Ax} + 2A - 5\cancel{Ax} - 5A + 4\cancel{Ax} = 8$$

$$-3A = 8$$

$$A = -8/3 \qquad y(x) = -\frac{8}{3}xe^x$$

(c) inhomog. prob. 2. $y(x) = (Bx + C)\sin x + (Dx + E)\cos x$

$$y'(x) = (Bx + C + D)\cos x + (-Dx - B - F)\sin x$$

$$y''(x) = (-Bx - 2D - C)\sin x + (-Dx + 2B - F)\cos x$$

some yucky algebra here

$$(-5C - 5D + 3E + 2B + (3D - 5B)x)\cos x$$
$$+ (3C - 2D + 5E - 5B + (5D + 3B)x)\sin x = 10x \sin x$$

$$-5C - 5D + 3E + 2B = 0$$
$$3D - 5B = 0$$
$$3C - 2D + 5E - 5B = 0$$
$$5D + 3B = 10$$

~~$B = \frac{-50}{289}$, $C = \frac{255}{289}$~~

~~$D = \frac{455}{289}$, $E = \frac{425}{289}$~~

$B = \frac{15}{17}$ $\quad C = \frac{-50}{289}$

$D = \frac{25}{17}$ $\quad E = \frac{455}{289}$

$y(x) = c_1 e^x + c_2 e^{4x} - \frac{8}{3}xe^x +$ ~~$\frac{1}{289}\left((-50x + 255)\sin x + (455x + 425)\cos x\right)$~~

$$+ \left(\frac{15}{17}x - \frac{50}{289}\right)\sin x + \left(\frac{25}{17}x + \frac{455}{289}\right)\cos x$$

5. Solve $\frac{d^2x}{dt^2} + \omega^2 x = F_0 \sin \omega t \quad x(0) = 0, \;\; x'(0) = 0.$

$r^2 + \omega^2 = 0 \qquad x(t) = C_1 \sin \omega t + C_2 \cos \omega t$ ← homog. solution.

Thus the inhomog. sol'n must be

$$x(t) = t(A \sin \omega t + B \cos \omega t)$$

$$x' = t(Aw \cos wt - Bw \sin wt) + (A \sin wt + B \cos wt)$$

$$= (A - tBw) \sin wt + (B + tAw) \cos wt$$

$$x'' = (-2Bw - Atw^2) \sin wt + (2Aw - Btw^2) \cos wt$$

$$x'' + \omega^2 x = F_0 \sin \omega t$$

$$(-2Bw - At\omega^2) \sin \omega t + (2Aw - Bt\omega^2) \cos \omega t + \omega^2 t A \sin \omega t + \omega^2 t B \cos \omega t = F_0 \sin \omega t$$

$$-2B\omega \sin \omega t + 2A\omega \cos \omega t = F_0 \sin \omega t$$

$$-2B\omega = F_0 \qquad 2A\omega = 0$$

$$B = \frac{-1}{2\omega} F_0 \qquad A = 0$$

$$x(t) = -\frac{1}{2\omega} F_0 \, t \cos \omega t$$

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General Solution is

$$x(t) = C_1 \sin \omega t + C_2 \cos \omega t - \frac{1}{2} F_0 t \cos \omega t$$

w/ $x(0) = 0$ & $x'(0) = 0$ we find $C_1 = \frac{F_0}{2\omega^2}$ $C_2 = 0$

so the sol'n is $x(t) = \frac{F_0}{2\omega^2} \sin \omega t - \frac{F_0}{2\omega} t \cos \omega t$

6. Find the general solution to the DE $4y'' + 36y = \sec 3x$.

Variation of Parameters

→ sol'n to corresponding homog. prob.

$4r^2 + 36 = 0 \qquad r^2 + 9 = 0 \qquad y(t) = c_1 \cos 3x + c_2 \sin 3x$

→ sol'n to inhom. prob.

$y(x) = u_1(x) \cos 3x + u_2(x) \sin 3x$

$y' = -u_1 3 \sin 3x + 3u_2 \cos 3x + \underbrace{u_1' \cos 3x + u_2' \sin 3x}_{=0}$

$y'' = -u_1 9 \cos 3x - u_1' 3 \sin 3x + u_2 9 \sin 3x + 3u_2' \cos 3x$

$4(-u_1 9 \cos 3x - u_1' 3\sin 3x - 9u_2 \sin 3x + 3u_2' \cos 3x) + 36(u_1 \cos 3x + u_2 \sin 3x)$

$= \sec 3x$

$-12u_1' \sin 3x + 12u_2' \cos 3x = \sec 3x$

$u_1' \cos 3x + u_2' \sin 3x = 0$

$$\begin{bmatrix} -12 \sin 3x & 12 \cos 3x \\ \cos 3x & \sin 3x \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} \sec 3x \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{-12} \begin{bmatrix} \sin 3x & -12 \cos 3x \\ -\cos 3x & -12 \sin 3x \end{bmatrix} \begin{bmatrix} \sec 3x \\ 0 \end{bmatrix}$$

$u_1' = -\frac{1}{12} \sin 3x \sec 3x = -\frac{1}{12} \frac{\sin 3x}{\cos 3x}$

$u_2' = \frac{-1}{12}(-\cos 3x) \sec 3x = \frac{1}{12}$

$u_1(x) = \frac{1}{12} \cdot \frac{1}{3} \ln|\cos 3x| \quad , \quad u_2(x) = \frac{1}{12} x$

$y(x) = \left(c_1 + \frac{1}{36} \ln|\cos 3x|\right) \cos 3x + \left(c_2 + \frac{1}{12} x\right) \sin 3x$

7. Consider the differential equation

$$t^2y'' - 2y = 3t^2 - 3, \qquad t > 0.$$

Suppose that the functions $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ satisfy the corresponding homogeneous equation. What is the general solution to the nonhomogeneous equation? **Hint:** Is the DE in standard form? (Hint only necessary if using Variation of parameter formula).

nonhomog. sol'n :

$$y(t) = u_1(t)t^2 + u_2(t)t^{-1}$$

$$y' = u_1 2t + u_2(-1)t^{-2} + \underbrace{u_1't^2 + u_2't^{-1}}_{\text{assume } 0}$$

$$y'' = u_1 2 + u_1'2t + u_2 2t^{-3} + u_2'(-1)t^{-2}$$

$$t^2y'' - 2y = 3t^2 - 3$$

$$\hookrightarrow t^2(2u_1 + 2tu_1' + 2t^{-3}u_2 - u_2't^{-2}) - 2(u_1t^2 + u_2t^{-1}) = 3t^2 - 3$$

$$2t^3u_1' - u_2' = 3t^2 - 3$$

$$t^2u_1' + \tfrac{1}{t}u_2' = 0 \rightarrow t^3u_1' + u_2' = 0$$

adding the eq'ns gives

$$3t^3u_1' = 3t^2 - 3$$

$$u_1' = \frac{3t^2-3}{3t^3} = \frac{1}{t} - \frac{1}{t^3} = t^{-1} - t^{-3}$$

so by eq'n #2 $\quad u_2' = -t^3u_1'$

$$= -t^3(t^{-1} - t^{-3})$$

$$= -t^2 + 1$$

$$\Rightarrow u_1(t) = \ln t + \frac{1}{2t^2}, \quad u_2(t) = -\frac{1}{3}t^3 + t$$

$$\text{so } y(t) = \left(\ln t + \frac{1}{2t^2} + c_1\right)t^2 + \left(-\frac{1}{3}t^3 + t + c_2\right)\cdot\frac{1}{t} = c_1t^2 + \frac{c_2}{t} - \frac{1}{3}t^2 + t^2\ln t + \frac{3}{2}$$