

## HOMEWORK 5

Please show all your work. When possible, write your answers in complete sentences. The easier your solution is to read, the easier it is to give you feedback and points.

1. A mass of 1 slug is suspended from a spring whose spring constant is 9 lb/ft. The medium through which the mass moves offers a damping force numerically equal to  $\sqrt{2}$  times the instantaneous velocity. The mass is initially released from a point 1 foot above the equilibrium position with an upward velocity of  $\sqrt{3}$  ft/s. Find the equation of motion.
2. A mass weighing 8 lb stretches a spring 1.5 in. The mass is also attached to a damper with a coefficient  $\gamma$ . Determine the value of  $\gamma$  for which the system is critically damped; be sure to give the units for  $\gamma$ .
3. For this problem we'll be considering a simplified version of the forced harmonic oscillator (starting from rest):

$$y'' + y = \sin \omega t, \quad y(0) = 0, \quad y'(0) = 0$$

where the forcing has amplitude one and frequency  $\omega$ .

- (a) Use the method of undetermined coefficients to solve the I.V.P. for  $\omega \neq 1$  (why does  $\omega \neq 1$  matter?).
  - (b) Use Mathematica to plot the solution when  $\omega = .5, \omega = .7, \omega = .9$ . Attach your plots to the assignment ( `Plot[{x, x^2, x^3}, {x,-3,3}]` would make graphs of three functions on the same axes; I suggest you plot these together and for  $0 \leq t \leq 60$ .) What happens to the solution as  $\omega$  gets closer to the natural frequency of the oscillator ( $\omega = 1$ )?
  - (c) Now solve the IVP for  $\omega = 1$ . Plot your solution on the same axes as the solution when  $\omega = .9$ . How are they related? How are they different? Your new solution should grow, in amplitude, without bound. This phenomenon is called resonance and can occur when a system is forced at, or close to, its natural frequency - even by a very small amount of forcing. (See <http://www.scienceclarified.com/everyday/Real-Life-Chemistry-Vol-4/Resonance.html> for some more, mostly correct, information about resonance.)
4. **Taylor Series Convergence:** Many important functions in mathematics and physics are not as simple as  $\sin(x)$  and  $e^x$ , but nevertheless arise in applications.

- (a) For instance

$$J_0(x) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} (m!)^2}$$

is known as the Bessel function of the first kind of order 0. It arises as one solution to the Bessel equation:  $x^2 y'' + xy' + (x^2 - \nu^2)y = 0$  when  $\nu = 0$ .

Use Mathematica to plot some partial sums on the interval  $0 \leq x \leq 10$  for partial sums up to order 24 (including terms up to degree 24). For example with  $\sin(x)$

```
pSums = Table[ Sum[ (-1)^k x^(2k+1)/(2k+1)!, {k,0,n} ], {n,0,10}]
```

```
Plot[ pSums, {x,-10,10} ]
```

Do this for the Bessel function above and attach your plot. Notice that when  $x$  is large the solution appears to oscillate ... does something about the differential equation (when  $\nu = 0$ ) suggest that this should be the case?

(b) Find the interval of convergence of the power series:  $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n (x+5)^n}{n}$

5. Solve the differential equation by means of a power series about the given point  $x_0$ . Find the recurrence relation; also find the first four terms in each of two linearly independent solutions (unless the series terminates sooner). If possible, find the general term in the solution:

$$y'' - xy' - y = 0, x_0 = 0$$

6. Solve the differential equation by means of a power series about the given point  $x_0$ . Find the recurrence relation; also find the first four terms in each of two linearly independent solutions (unless the series terminates sooner). If possible, find the general term in the solution:

$$2y'' + (x+1)y' + 3y = 0, x_0 = 3$$