Math 353 **Handout 5** **Spring 2010**

Homework 5

Please show all your work. When possible, write your answers in complete sentences. The easier your solution is to read, the easier it is to give you feedback and points.

1. A mass of 1 slug is suspended from a spring whose spring constant is 9 lb/ft. The medium through which the mass moves offers a damping force numerically equal to $\sqrt{2}$ times the instantaneous velocity. The mass is initially released from a point 1 foot above the equilibrium position with an upward velocity of $\sqrt{3}$ ft/s. Find the equation of motion.

$m = 1$ slug, $k = 9$ lb/ft, $\gamma = \sqrt{2}$ (don't need to solve)

$$my'' + \gamma y' + ky = 0$$

$$y'' + \sqrt{2}y' + 9y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

2. A mass weighing 8 lb stretches a spring 1.5 in. The mass is also attached to a damper with a coefficient γ. Determine the value of γ for which the system is critically damped; be sure to give the units for γ.

weight = mg

$8\text{ lb} = m\, 32\, \text{ft}/s^2$

$m = \dfrac{8\text{ lb}}{32\,\text{ft}/s^2} = \dfrac{1}{4}\text{ slug}$

Hooke's Law: $F = kx$

$8\text{ lb} = k\; 1.5\text{ in} * \dfrac{1\text{ ft}}{12\text{ in}}$

$8\text{ lb} = k * \dfrac{1}{8}\text{ ft}$

$k = 64\,\dfrac{\text{lb}}{\text{ft}}$

if y = position of mass

$$my'' + \gamma y' + ky = 0$$

$$\frac{1}{4}y'' + \gamma y' + 64y = 0$$

$$\frac{1}{4}r^2 + \gamma r + 64 = 0 \Rightarrow r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4 \cdot \frac{1}{4} \cdot 64}}{2 \cdot \frac{1}{4}}$$

critical damping is when discriminate is 0

$$\gamma^2 - 64 = 0$$

$\gamma = \pm 8$ (must be positive)

so $\underline{\underline{\gamma = 8}}$

3. For this problem we'll be considering a simplified version of the forced harmonic oscillator (starting from rest):

$$y'' + y = \sin \omega t, \quad y(0) = 0, \quad y'(0) = 0$$

where the forcing has amplitude one and frequency ω.

(a) Use the method of undetermined coefficients to solve the I.V.P. for $\omega \neq 1$ (why does $\omega \neq 1$ matter?).

Homog. solution : $y(t) = C_1 \sin t + C_2 \cos t$, if $\omega = 1$, then $\sin \omega t$ is a solution to the homog. eq'n

Particular solution to inhomog. eq'n ($\omega \neq 1$)

$$y(t) = A \sin \omega t + B \cos \omega t$$
$$y''(t) = -A\omega^2 \sin \omega t - B\omega^2 \cos \omega t$$

PLUG IN to DE

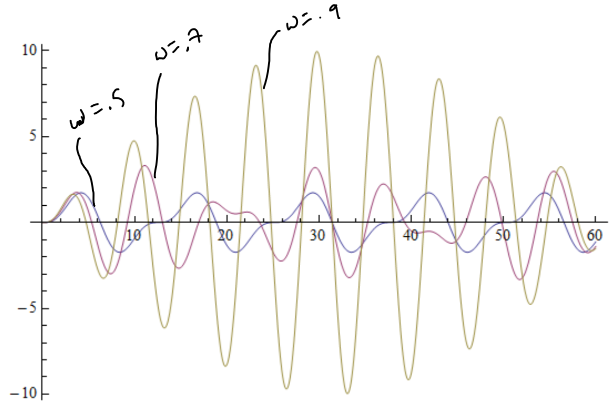
$$-A\omega^2 \sin \omega t - B\omega^2 \cos \omega t + A \sin \omega t + B \cos \omega t = \sin \omega t$$
$$A(1-\omega^2) \sin \omega t + B(1-\omega^2) \cos \omega t = \sin \omega t$$
$$\Rightarrow \quad A = \frac{1}{1-\omega^2}, \quad B = 0$$

Total general solution ($\omega \neq 1$) : $y(t) = C_1 \sin t + C_2 \cos t + \frac{1}{1-\omega^2} \sin \omega t$

Notice $y(0) = C_2$ & $y'(0) = C_1 + \frac{\omega}{1-\omega^2}$, using the initial conditions we get $C_2 = 0$, $C_1 = \frac{-\omega}{1-\omega^2}$

$$\Rightarrow \quad y(t) = \frac{-\omega}{1-\omega^2} \sin t + \frac{1}{1-\omega^2} \sin \omega t$$

(b) Use Mathematica to plot the solution when $\omega = .5, \omega = .7, \omega = .9$. Attach your plots to the assignment (Plot[$\{x, x^2, x^3\}$, $\{x$,-3,3$\}$] would make graphs of three functions on the same axes; I suggest you plot these together and for $0 \leq t \leq 60$.) What happens to the solution as ω gets closer to the natural frequency of the oscillator ($\omega = 1$)?



as $\omega \to 1^-$ the amplitude of the system response (the oscillations) gets larger.

(c) Now solve the IVP for $\omega = 1$. Plot your solution on the same axes as the solution when $\omega = .9$. How are they related? How are they different? Your new solution should grow, in amplitude, without bound. This phenomenon is called resonance and can occur when a system is forced at, or close to, its natural frequency - even by a very small amount of forcing. (See http://www.scienceclarified.com/everyday/Real-Life-Chemistry-Vol-4/Resonance.html for some more, mostly correct, information about resonance.)

particular solution $y(t) = t(A\sin t + B\cos t)$

↑ because $\sin t$ & $\cos t$ both solve homog. eq'n

$y(t) = At\sin t + Bt\cos t$

$y'(t) = At\cos t + A\sin t + Bt(-\sin t) + B\cos t = (At+B)\cos t + (-Bt+A)\sin t$

$y''(t) = (At+B)(-\sin t) + A\cos t + (-Bt+A)\cos t + (-B)\sin t$

$y''(t) = (At-2B)\sin t + (-Bt+2A)\cos t$

Plug in

$(\cancel{At}-2B)\sin t + (-\cancel{Bt}+2A)\cos t + \cancel{At\sin t} + \cancel{Bt\cos t} = \sin t$

$-2B\sin t + 2A\cos t = \sin t$

$-2B = 1, \quad 2A = 0$

$B = \frac{-1}{2}, \quad A = 0$

particular sol'n $y(t) = -\frac{1}{2}t\cos t$, general solution: $y(t) = c_1\sin t + c_2\cos t - \frac{1}{2}t\cos t$

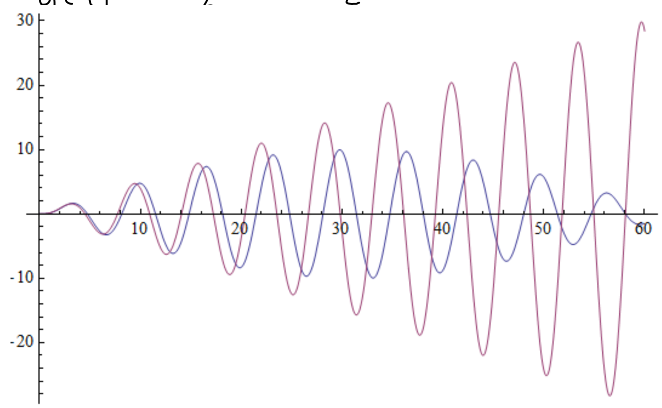
Sol'n to IVP: use $y(0) = 0$ & $y'(0) = 0$

$y(0) = c_2 = 0$

$y'(t) = c_1\cos t - c_2\sin t - \frac{1}{2}(t(-\sin t) + \cos t)$

$y'(0) = c_1 - \frac{1}{2} = 0 \Rightarrow c_1 = \frac{1}{2}$

$u(t) = \frac{1}{2}\sin t - \frac{1}{2}t\cos t$



4. **Taylor Series Convergence:** Many important functions in mathematics and physics are not as simple as $\sin(x)$ and e^x, but nevertheless arise in applications.

(a) For instance

$$J_0(x) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}$$

is known as the Bessel function of the first kind of order 0. It arises as one solution to the Bessel equation: $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ when $\nu = 0$.

Use Mathematica to plot some partial sums on the interval $0 \le x \le 10$ for partial sums up to order 24 (including terms up to degree 24). For example with $sin(x)$

```
pSums = Table[ Sum[ (-1)^k x^(2k+1)/(2k+1)!, {k,0,n} ], {n,0,10}]

Plot[ pSums, {x,-10,10} ]
```

Do this for the Bessel function above and attach your plot. Notice that when x is large the solution appears to oscillate ... does something about the differential equation (when $\nu = 0$) suggest that this should be the case?

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(b) Find the interval of convergence of the power series: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 2^n (x+5)^n}{n}$

ratio test:

$$\lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1} 2^{n+1} (x+5)^{n+1}}{n+1}\right|}{\left|\frac{(-1)^n 2^n (x+5)^n}{n}\right|} = \lim_{n\to\infty} \left|2 \frac{n}{n+1} (x+5)\right| = 2|x+5|$$

Need $2|x+5| < 1$

$|x+5| < \frac{1}{2}$ ($e = \frac{1}{2}$ is radius of conv)

so we have conv. on $-\frac{1}{2} < x+5 < \frac{1}{2}$ or

$-5.5 < x < -4.5$

at $x = -5.5$

$$\sum \frac{(-1)^n 2^n (-5.5+5)^n}{n} = \sum \frac{(-1)^n 2^n \left(\frac{-1}{2}\right)^n}{n} = \sum \frac{(-1)^n (-1)^n}{n} = \sum \frac{1}{n} \text{ diverges}$$

at $x = -4.5$

$$\sum \frac{(-1)^n 2^n (-4.5+5)^n}{n} = \sum \frac{(-1)^n 2^n \left(\frac{1}{2}\right)^2}{n} = \sum \frac{(-1)^n}{n} \text{ converges}$$

interval of convergence $(-5.5, 4.5]$

5. Solve the differential equation by means of a power series about the given point x_0. Find the recurrence relation; also find the first four terms in each of two linearly independent solutions (unless the series terminates sooner). If possible, find the general term in the solution:

$$y'' - xy' - y = 0, x_0 = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum^{\infty} a_n x^n = 0$$

5. Solve the differential equation by means of a power series about the given point x_0. Find the recurrence relation; also find the first four terms in each of two linearly independent solutions (unless the series terminates sooner). If possible, find the general term in the solution:

$$y'' - xy' - y = 0, x_0 = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\underbrace{2a_2 - a_0}_{=0} + \sum_{n=1}^{\infty} \big(\underbrace{a_{n+2}(n+2)(n+1) - na_n - a_n}_{=0} \big) x^n = 0$$

$$a_2 = \frac{a_0}{2}$$

$$a_{n+2}(n+1)(n+2) = na_n + a_n$$

$$a_{n+2} = \frac{\cancel{(n+1)} a_n}{\cancel{(n+1)}(n+2)}, \text{ for } n \geq 1$$

(works for $n=0$ too!)

$$a_3 = \frac{a_1}{3}$$

$$a_4 = \frac{a_2}{4} = \frac{1}{4} a_2 = \frac{1}{4} \cdot \frac{1}{2} a_0$$

$$a_5 = \frac{1}{5} a_3 = \frac{1}{5} \cdot \frac{1}{3} \cdot a_1$$

if n is even

$$a_n = a_{2k} = \frac{1}{2 \cdot 4 \cdots (2k)} a_0 = \frac{1}{2^k k!} a_0 \quad \left(\text{where } k = \frac{n}{2}\right)$$

if n is odd

$$a_n = a_{2k+1} = \frac{1}{1 \cdot 3 \cdot 5 \cdots 2k+1} a_1 = \frac{k! \, 2^k}{(2k+1)!} a_1$$

$$y(x) = a_0 \left(1 + \frac{1}{2} x^2 + \frac{1}{2 \cdot 4} x^4 + \frac{1}{2 \cdot 4 \cdot 6} x^6 + \cdots \right) + a_1 \left(x + \frac{1}{3} x^3 + \frac{1}{3 \cdot 5} x^5 + \frac{1}{3 \cdot 5 \cdot 7} x^7 + \cdots \right)$$

$$= a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!} + a_1 \sum_{k=0}^{\infty} \frac{2^k k!}{(2k+1)!} x^{2k+1}$$

6. Solve the differential equation by means of a power series about the given point x_0. Find the recurrence relation; also find the first four terms in each of two linearly independent solutions (unless the series terminates sooner). If possible, find the general term in the solution:

$$2y'' + \underbrace{(x+1)}_{}y' + 3y = 0, \quad x_0 = 3$$

$$y(x) = \sum_{n=0}^{\infty} a_n (x-3)^n$$

$$2\sum_{n=2}^{\infty} a_n n(n-1)(x-3)^{n-2} + \big((x-3)+4\big)\sum_{n=1}^{\infty} a_n n(x-3)^{n-1} + 3\sum_{n=0}^{\infty} a_n (x-3)^n = 0$$

$$2\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-3)^n + \sum_{n=1}^{\infty} a_n n(x-3)^n + 4\sum_{n=0}^{\infty} a_{n+1}(n+1)(x-3)^n + 3\sum_{n=0}^{\infty} a_n (x-3)^n = 0$$

$$\underbrace{4a_2 + 4a_1 + 3a_0}_{=0} + \sum_{n=1}^{\infty}\Big[\underbrace{2a_{n+2}(n+2)(n+1) + na_n + 4a_{n+1}(n+1) + 3a_n}_{=0}\Big](x-3)^n = 0$$

$$a_{n+2} = \frac{-4(n+1)a_{n+1} - (n+3)a_n}{2(n+2)(n+1)} \leftarrow$$ for $n \geq 1$, but also works for $n=0$.

$n=0$
$$a_2 = \frac{-4a_1 - 3a_0}{4} = -a_1 - \frac{3}{4}a_0$$

$n=1$
$$a_3 = \frac{-4(2)a_2 - 4a_1}{2(3)(2)} = \frac{-8a_2 - 4a_1}{12} = -\frac{2}{3}a_2 - \frac{1}{3}a_1 = -\frac{2}{3}\left(-a_1 - \frac{3}{4}a_0\right) - \frac{1}{3}a_1$$
$$= \frac{2}{3}a_1 + \frac{1}{2}a_0 - \frac{1}{3}a_1 = \frac{1}{3}a_1 + \frac{1}{2}a_0$$

$n=2$
$$a_4 = \frac{-4(3)a_3 - 5a_2}{2(4)(3)} = \frac{-12}{24}a_3 - \frac{5}{24}a_2 = \frac{-1}{2}\left[\frac{1}{3}a_1 + \frac{1}{2}a_0\right] - \frac{5}{24}\left[-a_1 - \frac{3}{4}a_0\right]$$
$$= \frac{-1}{6}a_1 - \frac{1}{4}a_0 + \frac{5}{24}a_1 + \frac{5}{32}a_0 = \frac{1}{24}a_1 - \frac{3}{32}a_0$$

$$y(x) = a_0 + a_1(x-3) + a_2(x-3)^2 + a_3(x-3)^3 + a_4(x-4)^4 + \cdots$$
$$= a_0 + a_1(x-3) + \left(-a_1 - \frac{3}{4}a_0\right)(x-3)^2 + \left(\frac{1}{3}a_1 + \frac{1}{2}a_0\right)(x-3)^3 + \left(\frac{1}{24}a_1 - \frac{3}{32}a_0\right)(x-3)^4 + \cdots$$
$$= a_0\left[1 - \frac{3}{4}(x-3)^2 + \frac{1}{2}(x-3)^3 - \frac{3}{32}(x-3)^4 + \cdots\right] + a_1\left[(x-3) - (x-3)^2 + \frac{1}{3}(x-3)^3 + \frac{1}{24}(x-3)^4 + \cdots\right]$$

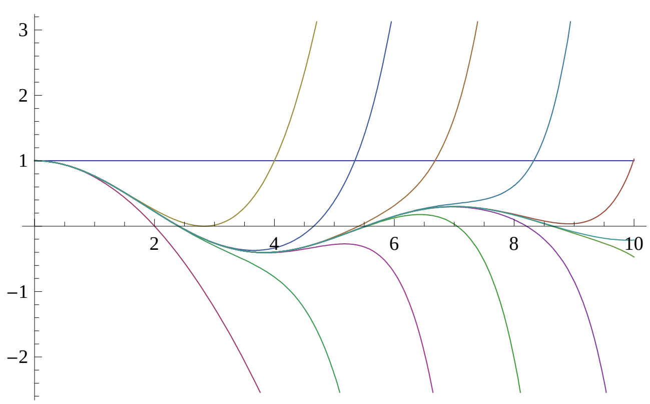
See the last page for some sample Mathematica to help w/ this rather error prone process...

In[93]:= `pSums = Table[1 + Sum[(-1)^(m) x^(2 m) / (2^(2 m) * (m!)^2), {m, 1, n}], {n, 0, 12}]`

Out[93]=
$$\left\{1,\ 1-\frac{x^2}{4},\ 1-\frac{x^2}{4}+\frac{x^4}{64},\ 1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304},\ 1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\frac{x^8}{147\,456},\right.$$
$$1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\frac{x^8}{147\,456}-\frac{x^{10}}{14\,745\,600},\ 1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\frac{x^8}{147\,456}-\frac{x^{10}}{14\,745\,600}+\frac{x^{12}}{2\,123\,366\,400},$$
$$1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\frac{x^8}{147\,456}-\frac{x^{10}}{14\,745\,600}+\frac{x^{12}}{2\,123\,366\,400}-\frac{x^{14}}{416\,179\,814\,400},$$
$$1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\frac{x^8}{147\,456}-\frac{x^{10}}{14\,745\,600}+\frac{x^{12}}{2\,123\,366\,400}-\frac{x^{14}}{416\,179\,814\,400}+\frac{x^{16}}{106\,542\,032\,486\,400},$$
$$1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\frac{x^8}{147\,456}-\frac{x^{10}}{14\,745\,600}+\frac{x^{12}}{2\,123\,366\,400}-\frac{x^{14}}{416\,179\,814\,400}+\frac{x^{16}}{106\,542\,032\,486\,400}-$$
$$\frac{x^{18}}{34\,519\,618\,525\,593\,600},\ 1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\frac{x^8}{147\,456}-\frac{x^{10}}{14\,745\,600}+\frac{x^{12}}{2\,123\,366\,400}-$$
$$\frac{x^{14}}{416\,179\,814\,400}+\frac{x^{16}}{106\,542\,032\,486\,400}-\frac{x^{18}}{34\,519\,618\,525\,593\,600}+\frac{x^{20}}{13\,807\,847\,410\,237\,440\,000},$$
$$1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\frac{x^8}{147\,456}-\frac{x^{10}}{14\,745\,600}+\frac{x^{12}}{2\,123\,366\,400}-\frac{x^{14}}{416\,179\,814\,400}+\frac{x^{16}}{106\,542\,032\,486\,400}-$$
$$\frac{x^{18}}{34\,519\,618\,525\,593\,600}+\frac{x^{20}}{13\,807\,847\,410\,237\,440\,000}-\frac{x^{22}}{6\,682\,998\,146\,554\,920\,960\,000},$$
$$1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\frac{x^8}{147\,456}-\frac{x^{10}}{14\,745\,600}+\frac{x^{12}}{2\,123\,366\,400}-\frac{x^{14}}{416\,179\,814\,400}+$$
$$\frac{x^{16}}{106\,542\,032\,486\,400}-\frac{x^{18}}{34\,519\,618\,525\,593\,600}+\frac{x^{20}}{13\,807\,847\,410\,237\,440\,000}-$$
$$\left.\frac{x^{22}}{6\,682\,998\,146\,554\,920\,960\,000}+\frac{x^{24}}{3\,849\,406\,932\,415\,634\,472\,960\,000}\right\}$$

In[94]:= `Plot[pSums, {x, 0, 10}]`

Out[94]=



```
In[102]:= a[n_] := (-4*(n-1)*a[n-1] - (n+1)*a[n-2]) / (2*n*(n-1))

In[96]:= a[0] = a0

Out[96]= a0

In[97]:= a[1] = a1

Out[97]= a1

In[104]:= Simplify[a[2]]
```

Out[104]= $-\frac{3\,a0}{4}-a1$

```
In[105]:= Simplify[a[3]]
```

Out[105]= $\frac{1}{6}\,(3\,a0+2\,a1)$

```
In[111]:= Simplify[a[4]]
```

Out[111]= $\frac{1}{96}\,(-9\,a0+4\,a1)$

```
In[124]:= Clear[X]
```

To see what the sum is like let X represent x - 3

```
In[125]:= S4 = Sum[a[k] X^k, {k, 0, 4}]
```

Out[125]= $a0+a1\,X+\frac{1}{4}\,(-3\,a0-4\,a1)\,X^2+\frac{1}{12}\,(-2\,(-3\,a0-4\,a1)-4\,a1)\,X^3+\frac{1}{24}\left(\frac{3}{4}\,(-3\,a0-4\,a1)+4\,a1\right)X^4$

```
In[128]:= S4 = Simplify[S4]
```

Out[128]= $\frac{1}{24}\,a1\,X\,(24-24\,X+8\,X^2+X^3)+a0\left(1-\frac{3\,X^2}{4}+\frac{X^3}{2}-\frac{3\,X^4}{32}\right)$

```
In[129]:= S4 /. X → (x - 3)
```

Out[129]= $a0\left(1-\frac{3}{4}\,(-3+x)^2+\frac{1}{2}\,(-3+x)^3-\frac{3}{32}\,(-3+x)^4\right)+$
$\frac{1}{24}\,a1\,\left(24-24\,(-3+x)+8\,(-3+x)^2+(-3+x)^3\right)\,(-3+x)$