Homework 7

Please show all your work. When possible, write your answers in complete sentences. The easier your solution is to read, the easier it is to give you feedback and points.

1. Find the Laplace transform of $f(t) = (3t+1)u_2(t)$.

$$f(t) = (3(t-2)+6+1)\,u_2(t) = (3(t-2)+7)\,u_2(t) = 3(t-2)u_2(t) + 7u_2(t)$$

$$\mathcal{L}\{3(t-2)u_2(t) + 7u_2(t)\} = 3\mathcal{L}\{(t-2)u_2(t)\} + 7\mathcal{L}\{u_2(t)\}$$

$$= 3e^{-2s}\cdot\frac{1}{s^2} + 7\frac{e^{-2s}}{s} = e^{-2s}\left[\frac{3}{s^2} + \frac{7}{s}\right]$$

2. Find $\mathscr{L}^{-1}\left\{\dfrac{e^{-3s}}{s(s^2+3)}\right\}$.

$$\frac{1}{s(s^2+3)} = \frac{a}{s} + \frac{bs+c}{s^2+3} \Rightarrow 1 = a(s^2+3) + (bs+c)s$$

$$1 = (a+b)s^2 + 3a + cs \Rightarrow \cancel{c=1,\ a=0,\ b} \quad 3a=1,\ c=0,\ a+b=0$$

$$\Rightarrow a = \tfrac{1}{3},\ b = -\tfrac{1}{3},\ c = 0$$

$$\mathcal{L}^{-1}\left\{e^{-3s}\left(\frac{1/3}{s} - \frac{\frac{1}{3}s}{s^2+3}\right)\right\} = \mathcal{L}^{-1}\left\{e^{-3s}\frac{\frac{1}{3}}{s} - \frac{1}{3}e^{-3s}\frac{s}{s^2+(\sqrt{3})^2}\right\}$$

$$= \frac{1}{3}u_3(t)\cdot 1 - \frac{1}{3}u_3(t)\cos(\sqrt{3}(t-3))$$

$$= \frac{1}{3}u_3(t)\left[1 - \cos(\sqrt{3}(t-3))\right]$$

3. Use the Laplace transform to solve the initial-value problems

(a) $y'' + 3y' + 2y = u_2(t), \quad y(0) = 0, \quad y'(0) = 1$

$$\xrightarrow{\mathcal{L}} s^2Y(s) - s\overset{0}{\cancel{y(0)}} - \overset{1}{\cancel{y'(0)}} + 3\left(sY(s) - \overset{0}{\cancel{y(0)}}\right) + 2Y(s) = \frac{e^{-2s}}{s}$$

$$(s^2+3s+2)Y(s) - 1 = \frac{e^{-2s}}{s}$$

$$Y(s) = \frac{1}{s^2+3s+2} + \frac{e^{-2s}}{s(s^2+3s+2)}$$

$$Y(s) = \frac{1}{(s+1)(s+2)} + \frac{e^{-2s}}{s(s+1)(s+2)}$$

$$= \frac{1}{s+1} - \frac{1}{s+2} + e^{-2s}\left(\frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2}\right)$$

$$\mathcal{L}^{-1}$$

$$\rightarrow y(t) = e^{-t} - e^{-2t} + u_2(t)\left(\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right)$$

(b) $y' + 2y = f(t), \quad y(0) = 0$ where $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$

$$f(t) = (1 - u_1(t))\,t = t - u_1(t)\,t$$
$$= t - u_1(t)\big((t-1)+1\big) = t - u_1(t)(t-1) - u_1(t)$$

So the D.E. is

$$y' + 2y = t - u_1(t)(t-1) - u_1(t)$$

$$\mathcal{L} \;\hookrightarrow\; sY(s) - \overset{0}{y(0)} + 2Y(s) = \frac{1}{s} - e^{-s}\left[\frac{1}{s^2} + \frac{1}{s}\right]$$

$$(s+2)Y(s) = \frac{1}{s} - e^{-s}\left[\frac{1}{s^2} + \frac{1}{s}\right]$$

$$Y(s) = \frac{1}{s(s+2)} - e^{-s}\left[\frac{1}{s^2(s+2)} + \frac{1}{s(s+2)}\right]$$

$$Y(s) = \frac{\frac{1}{2}}{s} - \frac{\frac{1}{2}}{s+2} - e^{-s}\left[\frac{\frac{1}{2}}{s^2} - \frac{\frac{1}{4}}{s} - \frac{\frac{1}{4}}{s+2} + \frac{\frac{1}{2}}{s} - \frac{\frac{1}{2}}{s+2}\right]$$

$$= \frac{\frac{1}{2}}{s} - \frac{\frac{1}{2}}{s+2} - e^{-s}\left[\frac{\frac{1}{2}}{s^2} + \frac{\frac{1}{4}}{s} - \frac{\frac{3}{4}}{s+2}\right]$$

$$\mathcal{L}^{-1} \;\hookrightarrow\; y(t) = \frac{1}{2} - \frac{1}{2}e^{-2t} - u_1(t)\left[\frac{1}{2}(t-1) + \frac{1}{4} - \frac{3}{4}e^{-2(t-1)}\right]$$

4. Consider the initial value problem:

$$y'' + 2y' + \cancel{(1760^2\pi^2 + 4)}y = \delta(t-\pi), \quad y(0) = 0, \quad y'(0) = 0$$

$$y'' + 2y' + (880^2\pi^2 + 1)y = \delta(t-\pi)$$

(a) Use Laplace transforms to solve the IVP.

Let $\beta = 880^2\pi^2 + 1$ (or $1760^2\pi^2 + 1$)

$$y'' + 2y' + \beta y = \delta(t-\pi)$$

$$s^2 Y(s) - s y(0) - y'(0) + 2(s Y(s) - y(0)) + \beta Y(s) = e^{-\pi s}$$

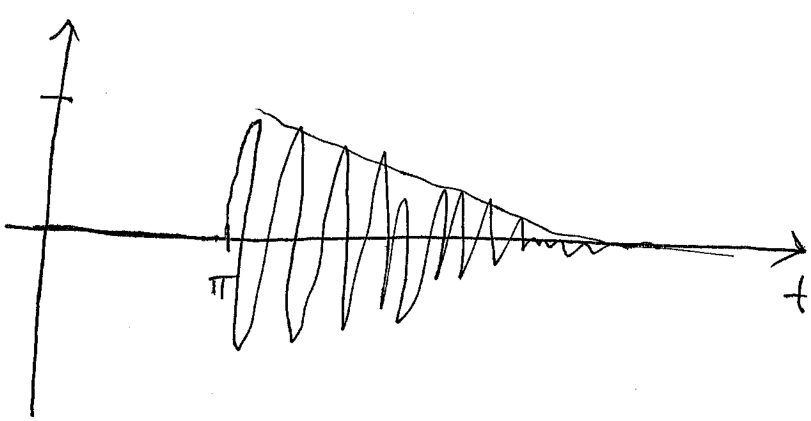
$$(s^2 + 2s + \beta) Y(s) = e^{-\pi s}$$

$$Y(s) = e^{-\pi s} \cdot \frac{1}{s^2 + 2s + 880^2\pi^2 + 1}$$

$$= e^{-\pi s} \cdot \frac{1}{(s+1)^2 + 880^2\pi^2} = \frac{1}{880\pi} \cdot e^{-\pi s} \frac{880\pi}{(s+1)^2 + 880^2\pi^2}$$

$$\hookrightarrow y(t) = \frac{1}{880\pi} e^{-(t-\pi)} u_\pi(t) \sin(880\pi(t-\pi))$$

$$y(t) = \frac{1}{880\pi} e^{-(t-\pi)} u_\pi(t) \sin(2\pi(440)(t-\pi))$$



(b) Use Mathematica to "play" your solution. For instance, the command below plays a standardized middle A note at 440 Hz for 10 seconds (the sound is very flat because it contains ONLY the pure harmonic, or sine wave):

```
Play[ Sin[ 440 * 2 * Pi * t ],{t,0,10}]
```

You'll probably have to make use of the UnitStep function to define your solution. For convenience, define a function as your solution, e.g.

```
y[t_] = Sin[ 440 * 2 * Pi * t];
```

(c) Once you've done (b) and defined a function as your solution, try this:

```
Play[ y[t] + y[t/2] + y[2t],{t,0,10} ]
```

What do you hear? How do the notes change? Musically, what is the difference iin the notes - do you know?

The notes differ by octaves.

(d) If you enjoy this sort of thing, try the command below, it plays the middle A note using a square wave instead of a sine wave:

```
Play[ SquareWave[ 440*t ], {t,0,10} ]
```

Why does the sound seem so different than when we play a pure sine wave? The basic frequency is the same.

The square wave is actually a superposition of sine waves of many different (unharmonious) frequencies.

(e) Hand-in a Mathematica printout showing your sound experiments.

5. Suppose that $F(s) = \mathscr{L}\{f(t)\}$ exists for $s > a \geq 0$. Show that is c if a positive constant, then

$$\mathscr{L}\{f(ct)\} = \frac{1}{c} F\left(\frac{s}{c}\right), \quad s > ca.$$

$$\mathcal{L}\{f(ct)\} = \int_0^\infty f(ct) e^{-st} dt \quad \text{let } u = ct, \quad (c>0)$$

$$du = c\,dt \qquad t=0 \Rightarrow u=0$$

$$\tfrac{1}{c} du = dt \qquad t \to \infty \Rightarrow u \to \infty$$

$$= \int_0^\infty f(u) e^{-su/c} \frac{1}{c} du = \frac{1}{c} \int_0^\infty f(u) e^{-\frac{s}{c}u} du$$

$$= \frac{1}{c} F\left(\frac{s}{c}\right) \quad \text{where } F(s) = \int_0^\infty f(t) e^{-st} dt$$

6. Find the eigenvalues and eigenfunctions for:

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0$$

$y = e^{ax} \rightarrow a^2 e^{ax} + \lambda e^{ax} = 0 \Rightarrow (a^2+\lambda) e^{ax} = 0$

$\Rightarrow a^2 = -\lambda \Rightarrow a = \pm\sqrt{-\lambda}$

Now if $\lambda < 0$ we get $a = \pm\sqrt{-\lambda}$ as real roots and have solution of the form $y(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x}$ that cannot satisfy the boundary conditions.

if $\lambda = 0$ we get a solution of the form $y = mx+b$, this satisfies the boundary conditions as long as $m = 0$. (any constant function is an eigenfunction)

So for $\lambda = 0$, $y(x) = 1$.

if $\lambda > 0$ $\quad y(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$

to satisfy the (Neumann) boundary conditions, $y'(0) = 0$ & $y'(L) = 0$

$y'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

$y'(0) = -c_1\sqrt{\lambda}(0) + c_2\sqrt{\lambda}\,1 = 0 \Rightarrow c_2 = 0$

$y'(L) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0$ so $c_1 = 0$ or

$\sin(\sqrt{\lambda}L) = 0 \Rightarrow \sqrt{\lambda} = n\pi$ for $n \in \mathbb{Z}$.

$\Rightarrow \lambda = n^2\pi^2$ for $n \in \mathbb{Z}$.

So for $n = 0, 1, 2, 3, \ldots$ the eigenpairs are

$\lambda_n = n^2\pi^2$ & $y_n(x) = \cos(n\pi x)$