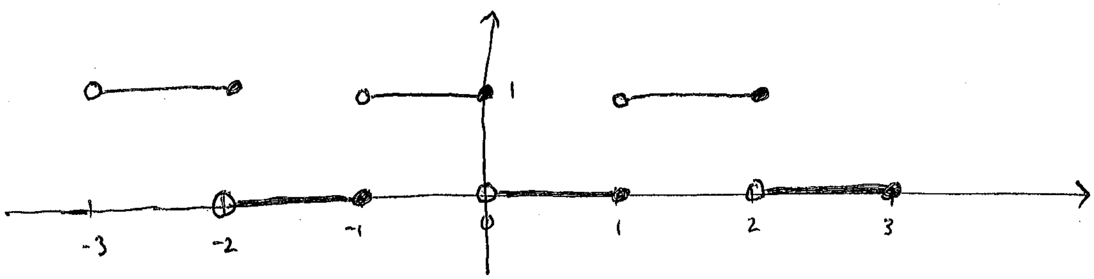
Math 353 Test 3 Name: Solutions Row:

READ FIRST. No calculators, books, notes or other written material allowed. Read each question very carefully. Show your work neatly for partial credit. To receive full credit your work should be correct, organized, include all supporting calculations, and clearly indicate your final answer. Correct answers with too few supporting calculations will receive little or no credit.

1. Consider the function $f(x) = \begin{cases} 1, & -1 < x \leq 0, \\ 0, & 0 < x \leq 1 \end{cases}$.

   (a) Suppose $f(x+2) = f(x)$. Sketch the graph of $f(x)$ for **three** periods.



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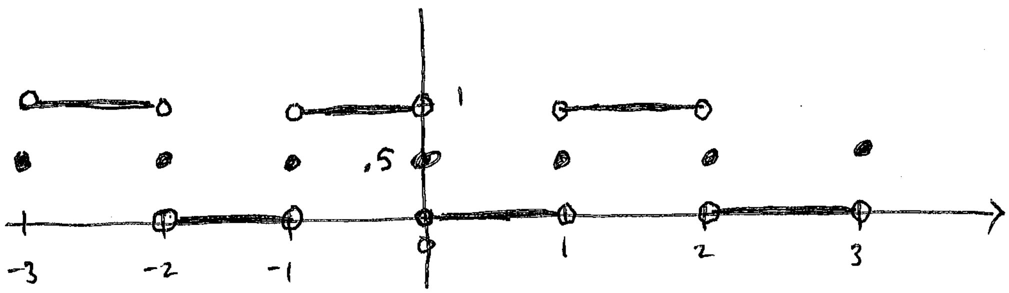
   (b) Sketch **three** periods of the function to which the Fourier series of $f(x)$ converges.



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2. By direct computation (using only the definition), compute $\mathscr{L}\{te^{2t}\}$.

$$\mathscr{L}\{te^{2t}\}(s) = \int_0^\infty te^{2t}e^{-st}\,dt = \int_0^\infty t\,e^{(2-s)t}\,dt$$

$$= \cancel{t\frac{1}{2-s}e^{(2-s)t}\Big|_0^\infty}^{\,0} - \frac{1}{2-s}\int_0^\infty e^{(2-s)t}\,dt$$

(goes to 0 if $s>2$)

$$= -\frac{1}{2-s}\cdot\frac{1}{2-s}\cdot e^{(2-s)t}\Big|_0^\infty = \frac{-1}{(2-s)^2}\Big[\underset{\substack{\uparrow \\ 0 \text{ if } s>2}}{e^{(2-s)\infty}} - 1\Big]$$

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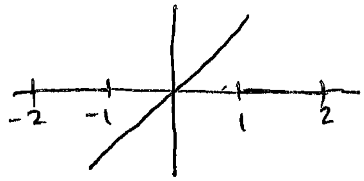
$$= \frac{1}{(2-s)^2} = \frac{1}{(s-2)^2} \quad \text{if } s>2.$$

3. For the nonperiodic function f (shown below) that is defined on $[0, 2)$, extend it as on odd function with period 4 and find an appropriate trig. series:

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & 1 \le x < 2 \end{cases}$$



$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right), \quad b_n = \frac{2}{2}\int_0^2 f(x) \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \int_0^1 x \sin\left(\frac{n\pi x}{2}\right) dx \overset{\text{parts} \dots}{=} \dots = \frac{4\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2} - \frac{2\cos\left(\frac{n\pi}{2}\right)}{n\pi}$$

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{4\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2} - \frac{2\cos\left(\frac{n\pi}{2}\right)}{n\pi} \right) \sin\left(\frac{n\pi x}{2}\right)$$

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You may use the Laplace transform table for the rest of the exam.

4. Compute the Laplace transform of

(i) $f(t) = (t-2)u_1(t)$

$= (t-1)u_1(t) - u_1(t)$

$F(s) = e^{-s} \cdot \frac{1}{s^2} - \frac{e^{-s}}{s}$

$(s>0)$

(ii) $g(t) = \begin{cases} 0, & 0 \le t < 3 \\ (t-3)^2 e^{t-3}, & t \ge 3 \end{cases}$

$\cancel{G(s) = \frac{2}{s^3} e^{-3s}}$

$G(s) = \frac{2}{(s-1)^3} e^{-3s} \quad (s>\phi)$

(iii) Find $\mathscr{L}\{e^{5t}t^3\}$

$\frac{3!}{(s-5)^4}$

$(s>5)$

(iv) $h(t) = \delta(t-6)\sin(2t)$

$\sin(12)e^{-6s}$

6 each

5. Find the inverse Laplace transform of $F(s) = \frac{s}{s^2-6s+13} = \frac{s}{s^2-6s+9+4}$

$= \frac{s}{(s-3)^2+2^2} = \frac{s-3+3}{(s-3)^2+2^2} = \frac{(s-3)}{(s-3)^2+2^2} + \frac{3}{2} \frac{2}{(s-3)^2+2^2}$

$f(t) = e^{3t}\cos(2t) + \frac{3}{2}e^{3t}\sin(2t)$

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6. Use the fact that $\dfrac{1}{s(s^2+1)} = \dfrac{1}{s} - \dfrac{1}{s^2+1}$ to solve the initial value problem

$$y'' + y = \begin{cases} 0, & 0 \le t < 2 \\ 1, & t \ge 2 \end{cases}, \quad y(0) = 0,\ y'(0) = 1$$

transform

$$s^2 Y(s) - s\,\cancel{y(0)}^{\,0} - \cancel{y'(0)}^{\,1} + Y(s) = \frac{e^{-2s}}{s}$$

$$(s^2+1)Y(s) - 1 = \frac{e^{-2s}}{s}$$

$$Y(s) = \frac{1}{s^2+1}\left(1 + \frac{e^{-2s}}{s}\right)$$

$$= \frac{1}{s^2+1} + e^{-2s}\frac{1}{s(s^2+1)}$$

$$= \frac{1}{s^2+1} + e^{-2s}\left[\frac{1}{s} - \frac{1}{s^2+1}\right]$$

inverse transform

$$y(t) = \sin(t) + u_2(t)\left[1 - \sin(t-2)\right]$$

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7. For the equation below, find the eigenvalues and eigenvectors:

$$y'' + \lambda y = 0, \qquad y(0) = 0, \qquad y'(L) = 0.$$

You may assume that $\lambda > 0$ since for $\lambda < 0$ the boundary conditions cannot be satisfied (unless $y(x) = 0$) and for $\lambda = 0$ the only solution is the trivial one.

Let $\lambda = \mu^2$ so $y'' + \mu^2 y = 0$

$r^2 + \mu^2 = 0 \rightarrow r = \pm\sqrt{-\mu^2} = \pm \mu i$

So $y(x) = a\cos\mu x + b\sin\mu x$

$y(0) = a = 0$ so a has to be zero.

Now we know

$y(x) = b\sin\mu x$

$y'(x) = b\mu\cos\mu x$, $y'(L) = b\mu\cos(\mu L) = 0$

So we need $\cos(\mu L) = 0$

which means μL must be an odd multiple of $\pi/2$ 16

$\mu L = (2n-1)\pi/2$, $n = 1, 2, 3, \ldots$

$\mu = \dfrac{(2n-1)\pi/2}{L} = \dfrac{(2n-1)\pi}{2L}$

eigenvalues are $\lambda_n = (\mu_n)^2 = \left(\dfrac{(2n-1)\pi}{2L}\right)^2$

& eigenfunctions $y_n(x) = \sin\left(\dfrac{(2n-1)\pi}{2L}x\right)$, $n = 1, 2, 3, \ldots$