Exam #1 Review Sheet – Practice Problems

1. Basic facts:
   a. State the fundamental counting principle in your own words.
   b. State the law of large numbers in your own words.
   c. State the complements principle in your own words.
   d. State the general (and special) additive principle in your own words.
   e. How many total subsets (of any size) can be selected from an n-item set? There are 2^n subsets total – this is related to the row sum of Pascal's triangle!

2. Consider the word *numbers*.
   a. How many different arrangements are there of the letters of the word *numbers*?
   b. How many arrangements begin with a vowel?
   c. How many arrangements have the word *numb* embedded in them?

   a. There are 7 letters, so there are 7! = 5,040 arrangements of the letters in *numbers*.
   b. There are 2 x 6! = 1,440 arrangements that begin with a vowel.
   c. The word *numb* can be treated as one block, so that we are just attempting to place four blocks. There are 4! = 24 ways to do this.

3. A five-card hand is dealt from a standard deck consisting of 52 cards (13 kinds and 4 suits).
   a. How many different hands are possible? C(52,5)
   b. In how many possible ways can a full house (3 of one kind, two of another kind) be dealt?

      P(13,2) = 156 = number of ways to choose the two face values. Order matters here because, for example, 3 kings and 2 jacks is different from 3 jacks and 2 kings.

      Now C(4,3) = 4 is the number of sets of three of the first face value, and C(4,2) = 6 is the number of sets of two of the second face value.

      Altogether, this gives: P(13,2) x C(4,3) x C(4,2) = 3,744 different possible full house hands.
   c. How about five cards of the same suit if exactly one of them is an Ace?

      Choose ace: 4 ways (this also determines the suit to be used for the hand)
      Choose four more cards from that suit: C(12,4)
      Answer is therefore: 4 x (12 x 11 x 10 x 9 / 4!) = 1,980 ways.

4. Consider a 7-question true and false test. Answer these questions *without* listing all the possibilities.
   a. How many different ways can a student answer the exam. $2^7 = 128$
   b. What is the *probability* that the student will get at least 5 correct?

      Cannot be answered unless we assume they are guessing randomly! In that case:
      P(At least 5 correct) = P(5 correct) + P(6 correct) + P(7 correct)
      = C(7,5)/128 + C(7,6)/128 + C(7,7)/128 ~= 22.7%

5. Twelve siblings (5 boys and 7 girls) must complete the Saturday chore list. There are five chores: rake the lawn, wash the dishes, clean the windows, vacuum the rug, and scrub the kitchen floor.

   In how many ways can five siblings be chosen to do these tasks?
      Order matters: P(12,5) = 12 x 11 x 10 x 9 x 8 = 95,040 ways.
   In how many ways can all girls be chosen for these chores?
      Order matters: P(7, 5) = 7 x 6 x 5 x 4 x 3 = 2,520 ways.

6. There are 5 statisticians and 6 economists to choose from.
   a. In how many ways can you select a committee consisting of 2 statisticians and 2 economists?
      C(5,2)xC(6,2) = 10 x 15 = 150 ways.
   b. What is the probability that at least one statistician is chosen for the committee (assuming they are randomly selected)?
      P(At least one statistician) = 1 – P(no statisticians) = 1 – C(6,4)/C(11,4) = 1 – 15/330 =~ 95.5%

Probability and Expected Value Problems (with some counting mixed in):

7. Basic facts:
   a. State the general multiplicative principle in your own words.
   b. Define the words “independent events” and “mutually exclusive events” in your own words.
   c. How are odds different from probability? How do you convert from one to the other?
   d. What is a ‘probability distribution,’ or ‘the distribution of a random variable’?
   e. Explain how to use the probability distribution to calculate the expected value of a random variable.

8. Timothy believes that if two fair coins are tossed then the probability of getting two heads should be 1/3 because there are only three possibilities: two heads, one head, or no heads. What is the correct probability? How might the law of large numbers come into play in helping Tim understand?

   The sample space consists of four outcomes: HH, HT, TH, and TT. The probability of HH is therefore ¼, not 1/3. By conducting an experiment with a large number of trials, Tim could compute the empirical probability which we know will tend to approach the theoretical probability in the long run. That should convince Tim that his reasoning is incorrect.

9. In a carnival game, you roll a pair of dice and add the values together. If the sum of the dice is anything **except** {5, 6, 7, 8, or 9}, you win $1. If one of those sums comes up, you must pay $1. Is this game fair, or does it favor either you or the carnival? Support your answer with appropriate calculations.

   Refer to the table in #10 to construct the probability distribution below:

Sum	2	3	4	5	6	7	8	9	10	11	12
Prob	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36
Win$1?	Y	Y	Y	-	-	-	-	-	Y	Y	Y

The probability of rolling a sum that would win you $1 is 12/36 (obtained by adding the relevant probabilities from the table above), which equals 1/3. So the expected winnings are about $0.33. But the probability of rolling a sum that would make you pay $1 is 24/36 = 2/3. So the expected losses are about $0.67. This means the game favors the carnival by a 2:1 margin!

10. A red die and a blue die are rolled.

S = { (1,1),(1,2),(1,3),(1,4),**(1,5)**,(1,6)
(2,1),(2,2),(2,3),(2,4),**(2,5)**,(2,6)
(3,1),(3,2),(3,3),(3,4),**(3,5)**,(3,6)
(4,1),(4,2),(4,3),(4,4),**(4,5)**,(4,6)
(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)
(6,1),(6,2),(6,3),(6,4),**(6,5)**,(6,6) }

   a. What is the probability that the sum of the two dice is at least 8?

There are 15 outcomes with a sum of 8 or more. So P = 15/36.

b. What is the probability that at least one of the dice is showing a 5?

These are shown in the sample space above. Counting them, we see there are 6 + 6 – 1 = 11. So P = 11/36.

We can view this as an OR problem:

P(5 on red OR 5 on blue) = P(5 on red) + P(5 on blue) – P(5 on both)
= 6/36 + 6/36 – 1/36 = 11/36.

c. What is the probability that the sum of the two dice is at least 8 given that one die is "5"?

"Given one of them is showing a 5" means the conditional sample space consists of just 11 items (shown in bold in the table above). Of these, 7 have a sum of 8 or more. So P = 7/11.

d. Are the events "at least one die is showing a 5" and "the sum of the dice is at least 8" mutually exclusive? Independent? Both? Neither? How do you decide?

They are neither. They are not mutually exclusive because they have common outcomes (like 5,4). They are not independent because the answers to parts a and c are different, so P(C) is not equal to P(C | A), which is required for independence.

11. Three cards are drawn without replacement from an ordinary deck of 52 playing cards.

a. What is the probability that the third card is a spade given the first two cards were spades?

Since the first two cards were spades, on the third draw there are only 11 spades in the deck and only 50 cards. $P(Spade_3 | spade_1 \text{ and } spade_2) = 11/50$

b. What is the probability that three spades will be selected in a row?

Method 1:

P(spade and spade and spade) = 13/52 x 12/51 x 11/50 = 1716/132,600 = 143/11,050 or about 0.013

Method 2: P(three spades in a row) = C(13,3) / C(52,3) = 286 / 22100 =~ 0.013.

12. There are two spinners: One is equally likely to land on 0, 1, 2, or 3 and the other is equally likely to land on 2, 3, 4, or 5.

a. Use a systematic list to show the set of all possible outcomes of the two spinners. What do we call this set?

It's called the sample space:

S = { (0,2), (0,3),(0,4), (0,5),
(1,2),(1,3),(1,4),(1,5),
(2,2),(2,3),(2,4),(2,5),
(3,2),(3,3),(3,4),(3,5)}

Let x equal the *sum* of the two spinners. Define the following events:

Let A be "x is odd."
Let B be "$x = 4$."
Let C be "the spinners match."
Let D be "$x > 5$."

b. Which pairs of events are mutually exclusive?

(A and B), (A and C), and (B and D) are all mutually exclusive pairs of events.

c. Find the following probabilities.

i.	P(A or B)	8/16 + 3/16 - 0/16 = 11/16
ii.	P(A or D)	8/16 + 6/16 – 2/16 = 12/16 = 3/4
iii.	P(A given D)	2/6 = 1/3
iv.	P(A and D)	2/16
v.	P(C given D)	1/6
vi.	P(C and D)	1/16

d. Are A and B independent? Are C and D independent? Explain how you know.
A and B are not independent – they are mutually exclusive, so knowing A has occurred drops the probability of B down to 0. To verify, we see P(A) = 8/16, but P(A | B) = 0/3. Because these are different, the events are not independent.
To check the independence of C and D, we check whether P(C | D) = P(C). We know P(C | D) = 1/6 from above. P(C) = 2/16 = 1/8. Because 1/6 is not equal to 1/8, these are **not** independent events.

13. A coin is tossed six times in succession.

a. What is the probability distribution of the number of heads in six tosses?

# heads	0	1	2	3	4	5	6
probability	1/64	C(6,1)/64 = 6/64	C(6,2)/64 = 15/64	C(6,3)/64 = 20/64	C(6,4)/64 = 15/64	6/64	1/64

b. What is the probability that

i. at least one head occurs?

Method 1: P(at least one head) = 1 – P(no heads) = $1-\frac{1}{2^6}=1-\frac{1}{64}=\frac{63}{64}$

Method 2: P(at least one head) = 1 – P(no heads) = 1 – C(6,0)$(1/2)^0(1/2)^6$ = 63/64.
(Method 2 uses the binomial probability formula)

ii. exactly three heads occur?

Method 1: P(exactly 3 heads) = $\frac{C(6,3)}{2^6}=\frac{20}{64}=0.3125$ (this was calculated in part a)

Method 2: P(exactly 3 heads) = C(6,3) x $(1/2)^3(1/2)^3$ ← that's from the binomial prob. formula

iii. more than four heads occur?

P(more than 4 heads) = P(5 heads) + P(6 heads) = 6/64 + 1/64 = 7/64 =~ 10.9% ← that's using the table in part a to get the probabilities).
This can also be done using the binomial probability formula, similar to a and b, above.

14. There are 4 gray, 6 pink, and 2 violet marbles in a hat. You pick 2 marbles from the hat. Marbles are <u>not</u> returned after they have been drawn. Find the probability that exactly one of the marbles is pink.

P(exactly one is pink) =
= P(pink on 1st and not pink on 2nd) + P(not pink on 1st and pink on second)
= 6/12 x 6/11 + 6/12 x 6/11
= 2(6/12 x 6/11)
= 72/132 or about 0.55.

15. A game is played in which a ball is selected from one of two boxes. Box 1 contains 9 red balls and 1 white ball. Box 2 contains 4 red balls and 1 white ball. To play the game, players spin spinner to select a box: Box 1 has an 80% chance of being selected and Box 2 has only a 20% chance. Then the player reaches into the box and draws one ball at random.

a. What is the probability of selecting a white ball in this game? 0.8 x 1/10 + 0.2 x 1/5 = 0.12

b. What is the probability of selecting a red ball in this game? 0.8 x 9/10 + 0.2 x 4/5 = 0.88

c. If a white ball is worth 10 points and a red ball is worth 2 points, what is the expected value for this game? 10x0.12 + 2x0.88 = 2.96 points per game