Math 151 Review - All Chapters Name:________________________________

1. **Chapter 1**

   (a) Complete the following definition: "Two sentences are equivalent if and only if..."

   (b) Short Essay: Compare and contrast the terms "equal," "equivalent," and (from Section 3.1) "logically equivalent."

   (c) Generalize from the following examples *one* common fact about mathematics. Express your generalization using placeholders.

   $1-(-2)=1+2$ $\quad y-(-1)=y+1$ $\quad -2-(-x)=-2+x$

   (d) What do identities express equivalence of?

   (e) Write a theorem that demonstrates how to solve algebra problems like $3x+4=13$. Identify the problem-pattern and solution-pattern in your theorem.

   (f) Which of the following equations are true generalizations?

   i. $\frac{x^2+x}{x}=x+1$

   ii. $\sqrt{x^2+4}=x+2$

   iii. $-3x^2=9x^2$

   iv. $(x^2)^3=x^5$

   v. $x^2x^5=x^7$

   vi. $\frac{x}{x^2+x}=\frac{1}{x}+1$

   vii. $\frac{x}{x^3+1}=\frac{1}{x^2+1}$

   viii. $(a+b)^2=a^2+b^2$

   ix. $(a+b)(a-b)=a^2-b^2$

   x. $a(b+c)=ac+bc$

   (g) Parentheses are used to group expressions in mathematics. Name two other ways.

   (h) Practice the "grammar" problems in the homework (e.g. A13-A24 in section 1.5, among others).

2. **Chapter 2**

   (a) Express the following in interval notation.

   i. $\{x\,:\,2<x \text{ and } x\le 5\}$ $\quad$ ii. $\{x\,:\,2<x \text{ or } x<-1\}$ $\quad$ iii. $\{x\,:\,-2\le x\}$

   (b) Write the definitions of $S\subset T$, $S\cup T$ and $S\cap T$. Illustrate each using Venn diagrams *and* numberlines.

   (c) Write the definition of set equality, and illustrate it with a specific example.

   (d) What is the complement of $(1,2)\cup[3,5)$? Of $(-\infty,2]\cup(\,(1,10)\cap[2,12]\,)$?

   (e) Let $f(x)=2x-1$, and let $g(x)=1-x$. Evaluate:

   i. $f(y)$ $\quad$ ii. $g(h+1)$ $\quad$ iii. $g(f(x))$ $\quad$ iv. $f(f(x))$

   (f) Express $f(x)=2x-1$ as a composition of two simpler functions.

   (g) Name two rules of algebra which might produce extraneous solutions. Give one example to illustrate when this might happen.

   (h) Under what conditions might a solution be dropped? Give an example to illustrate.

   (i) Solve $x=\sqrt{3x+10}$ using the rule on squaring.

   (j) Solve $x(x+1)=x(3x-2)$ using the rule on canceling.

   (k) Suppose 100 feet of fence are used to create three side-by-side-by-side rectangular animal pens, all of equal size, and suppose the pens are built along a garage wall so that no fence is needed along that side (see p. 144). Set up a formula for answering the question "What dimensions would maximize the area?

3. **Chapter 3**

   (a) Sketch truth tables for each of the following basic logic terms: not, and, or, implies, iff.

   (b) What is a conditional sentence?

   (c) Write out a complete truth table for (A and B) or [(not A) and (not B)]. Is it a tautology?

   (d) Prove (by using a truth table) that $A \Longrightarrow B$ is not logically equivalent to its converse.

   (e) How do you prove a logical statement is a tautology? How do you prove a logical statement is a contradiction?

   (f) Restate using the theorem on cases: $x\ge 8$ or $x\le -8 \Longrightarrow x^2\ge 64$.

   (g) Restate using the theorem on hypothesis in the conclusion: "If $S\subset T$, then $x\in S \Longrightarrow x\in T$."

(h) What is the negation (in positive form) of $H \Longrightarrow C$?

(i) State DeMorgan's Laws.

(j) Restate using the theorem on "or" in the conclusion: $x + y > 2 \Longrightarrow x > 1$ or $y > 1$.

(k) Work through problems B7 thru B12 on page 200.

(l) Work through problems B15 thru B21, and B22, on page 207.

4. **Chapter 4**

(a) What are the universal quantifier and the existential quantifier? Use each one in its own mathematical sentence.

(b) Short essay: What is the difference between a bound variable and a free variable? What do we usually call bound variables in Mathematics?

(c) Illustrate the difference between an unknown and a parameter by using one or more well-chosen examples. How do you know the difference?

(d) Work through exercises A2 through A12 on page 219.

(e) Explain why it is considered correct in Mathematics to say that the string *aaabacd* "has some b's in it."

(f) Which is easier to prove: a true generalization or a true existential statement? Explain.

(g) Is it possible to prove a false generalization? Is it possible to prove a generalization is false? (Hint: the answer to one of them is "yes", and the answer to the other is "no") Explain.

(h) Disprove $\{1, 2, 3\} = [1, 3]$. (See A5 through A26, p. 247, for similar problems).

(i) Use the Quadratic Theorem to solve for y: $x^2y + xy^2 + 2(x + 1) = 0$.

(j) One way to state the definition of "f is an <u>increasing</u> function" is "$f(x) < f(z)$ for all $x < z$.

   i. Restate the definition as a conditional statement.

   ii. Suppose f is *not* an increasing function. Write down precisely what this means by negating the definition of "f is an <u>increasing</u> function."

(k) Definition (that we have not used before): p is an <u>interior point</u> of the set S iff there exists an open interval (a, b) such that $p \in (a, b)$ and $(a, b) \subset S$.
Directions for the following conjectures: If the conjecture is true, prove it. Otherwise, explain why it is false. (Study note: why aren't these directions reversed?)

   i. Conjecture: 7 is an interior point of $[6, 8]$

   ii. Conjecture: 7 is an interior point of $\{7\}$

   iii. Conjecture: 3 is an interior point of $[3, 14)$

   iv. Conjecture: 4 is an interior point of $(3, 14)$

   v. Conjecture: 1 is an interior point of $(0.9, 1.1)$

(l) Let $f(x, y) = 2x + 3y$. Solve for x: $f(1, x) = f(2, -1)$.

(m) Definition (of a term you won't see until Calculus): The derivative of $a \sin(bx)$ is $ab \cos(bx)$.

   i. What is the derivative of $2 \sin(5x)$?

   ii. What is the derivative of $\sin(\frac{x}{2})$?

(n) Give the *form* of "$x > 5$ or $x < -5 \Longrightarrow |x| > 5$." Restate it using the theorem on cases. Finally, give the *form* of the negation of each version.

(o) Give the negation (in positive form) of "If $x^2 > y^2$, then $x > y$."

5. **Chapter 5**

(a) Give the form of the following proof, and explicitly state what has been proven.

$$\begin{aligned} \underline{\text{Proof:}}\quad S \subset T &\Longrightarrow (x \in S \Longrightarrow x \in T) && \text{(def'n of subset)} \\ &\Longrightarrow (x \notin T \Longrightarrow x \notin S) && \text{(contrapositive)} \\ &\Longrightarrow (x \in T^c \Longrightarrow x \in S^c) && \text{(def'n of complement, twice)} \\ &\Longrightarrow T^c \subset S^c && \text{(def'n of subset).}\quad \text{Q.E.D.} \end{aligned}$$

(b) Explain how to prove statements having each of the following forms: (i) $H \Longrightarrow C$, (ii) $A \iff B$, and (iii) $H \Longrightarrow (B$ or $C)$.

(c) Prove: $x < y \Longrightarrow 3x + 8 < 3y + 8$. For more, see 5.1(A3-A8, B14-B17), and 5.2(A42-A43).

(d) Prove: "$\forall x,\ x^2 > x$" is false. For more, see 5.2(A19-A40).