Math 151 Review - Chapter 4 Name: Jon's Answers (there may be others!)

1. **Open Sentences, Generalizations, and Existence Statements:** Classify each of the following, and write any quantifiers explicitly.

   (a) "$2x + 4 = 2(x + 2)$." generalization. quantifier is "$\forall x$".

   tricky! (I didn't mean to be tricky...)

   (b) "$ax^3 + bx^2 + cx + d = 0$ has a solution." (This looks like a generalization but happens to have a catch!) Open sentence about an existence statement. (quantifier: "$\exists x$"). Note: if $a \neq 0$ this would actually be a generalization (because then it is always true). In that case, the variables a, b, c, and d are universally quantified by "$\forall$".

   (c) "$x > 5 \Longrightarrow x^2 > 25$." generalization: "$\forall x$"

   (d) "($ax = ay$ and $a \neq 0$) $\Longrightarrow x = y$." generalization: "$\forall a$" (You should also add that x and y are universally quantified ("$\forall x, \forall y$"), though the "$\forall a$" is really what this sentence is all about.)

   (e) "$2x + 3 = 3x + 2$ has a solution." existence: ("$\exists x$")

2. **Compound Sentences.** Identify the three sentences inherent in the following statement. How do the roles of the variables change as your focus changes from examining the two "sub-sentences" to examining the compound sentence as a whole?

   $$\overbrace{\underbrace{x + a = b}_{①} \Longleftrightarrow \underbrace{x = b - a}_{②}}^{③}$$

   ① and ② are both open sentences in which x is unknown and a and b are parameters. The compound sentence ③ is about operations and order (solving equations), and the roles of x, a, and b change. Here, they serve as placeholders that demonstrate the relevant patterns.

3. **Directions:** How do the following sentences differ in terms of the intended interpretation? Which quantifiers are implied in each case?

   (a) "Solve: $x^2 + 2x + 1 = 0$." ← "Solve" suggests an existential statement.

   (b) "True or False: $x^2 + 2x + 1 = 0$." ← "T/F" suggests a generalization (in this case, a false one, because it is not true "$\forall x$".)

4. **Disproving Generalizations:** Explain how you disprove a generalization. Hint: What type of sentence is the negation of a generalization?

   The negation is an existence statement: you prove that a counterexample exists (by finding one and showing that it "works").

5. **Proving Existence Statements:** Explain how you prove an existential statement. Compare your response with the previous question.

   You just find an example and show that it "works." This is very similar to disproving a generalization.

6. **Disproving Generalizations:** The following are to be interpreted as generalizations. For each of them, (i) state the negation and (ii) find a counterexample that disproves the generalization. Be sure to demonstrate that your counterexample makes the negation true! I'll be looking for this level of justification on the exam.

   (a) $x^2 > 25 \Longrightarrow x > 5$.
   (i) "$\exists x$ s.t. $x^2 > 25$ but $x \leq 5$."
   (ii) Let $x = -10$. $(-10)^2 > 25$ but $-10 \leq 5$.

   (b) $x < 5 \Longrightarrow x^2 < 25$.
   (i) "$\exists x$ s.t. $x < 5$ and $x^2 \geq 25$."
   (ii) Let $x = -10$. $-10 < 5$ but $(-10)^2 \geq 25$.

(c) $a < b \Longrightarrow ax < bx$. (i) "$\exists$ a, b, and x s.t. $a<b$ but $ax \geq bx$."
(ii) Let $a=1, b=2, x=0$. $1<2$ but $1(0) \geq 2(0)$.

(d) $(x+y)^2 = x^2+y^2$. (i) "$\exists$ x and y s.t. $(x+y)^2 \neq x^2+y^2$."
(ii) Let $x=1$ and $y=1$. $(1+1)^2 \neq 1^2+1^2$; i.e. $4 \neq 2$.

7. **Using Definitions:** State the negation in positive form. You may need to expand the statement using definitions. **Note: I will provide the following definitions on the exam!** You should become familiar with their meaning, however.

- b is an upper bound for S iff $x \in S \Longrightarrow x \leq b$.
- S is bounded above iff there exists b such that b is an upper bound for S.
- f is an increasing function iff $x < y \Longrightarrow f(x) < f(y)$.

Note: I have not really shown my work here (I did it in my head). You should rewrite the sentence using the definitions and then find the negation.

(a) $f(x) = 1 - x$ is an increasing function. Negation: "$f(x) = 1-x$ is not increasing."
Negation in positive form: "$\exists$ x, y s.t. $x<y$ and $1-x \geq 1-y$."

(b) The set $\{1, 2, 3, ...\}$ is bounded above. Negation: "The set $\{1,2,3,...\}$ is not bounded above."
Negation in positive form: "$\forall b$, $\exists x$ s.t. $x \in \{1,2,3,...\}$ and $x \geq b$."

(c) $S \subset T$. Negation: $S \not\subset T$.
Negation in positive form: "$\exists$ x s.t. $x \in S$ and $x \notin T$."
(Note: the original statement can be expressed: "$\forall x$, $x \in S \Rightarrow x \in T$.")

8. **Applying Theorems:** (See p. 261.) The quadratic formula reads $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. In fact, the quadratic formula is the solution-pattern to a certain type of problem.

(a) Write the problem-pattern for the quadratic formula.
"Solve $ax^2+bx+c=0$," or simply "$ax^2+bx+c=0$."
this word is optional.

(b) Use the quadratic formula to solve for x: $x - 3x^2 = -4$.
$\Leftrightarrow -3x^2 + x + 4 = 0$
a is -3, b is 1, c is 4.
Solution: $x = \dfrac{-1 \pm \sqrt{1^2 - 4(-3)(4)}}{2(-3)}$

(c) Use the quadratic formula to solve for x: $y^2 + 2xy + x^2 = 0$.
$\Leftrightarrow x^2 + (2y)x + (y^2) = 0$
a is 1, b is $2y$, c is y^2.
Solution: $x = \dfrac{-(2y) \pm \sqrt{(2y)^2 - 4(1)(y^2)}}{2(1)}$

(d) Use the quadratic formula to solve for b: $2b^2 + bx + c = 0$.
a is "2" c is "c"
b is "x"
Solution: $b = \dfrac{-(x) \pm \sqrt{x^2 - 4(2)(c)}}{2(2)}$

9. **Other things you should know.** This review sheet is not comprehensive. Here are a few other things I expect you to be able to do:

- Identify placeholders vs. free variables, and unknowns vs. parameters.
- Know definitions of terms such as $S \subset T$, $S \cap T$, $S \cup T$. (See p. 262, Ex. 3, to see what I mean).
- Use symbols such as $\exists$, $\forall$, s.t., etc., to appropriately write concise mathematical statements. *Please use as many quantifiers as are necessary to make the meaning clear! Friday's quiz was unique, in that I told you ahead of time not to write $\forall x$ when it was clearly implied. On the exam, I'd like you to show me that you know it is supposed to be there!*

Read This! (*)

- And so on... let the homework and old exams (available at Cards'n'Copies) be your guide.