M151 Review – A partial answer key

Chapter 1:

(a) Two sentences are equivalent iff they are true for the same values of the variable and false for the same values of the variable.

(b) The term "equal" usually is used to relate expressions (eg. $x+1=2$, or $x+1=1+x$). The term "equivalent" is more general, and can refer to equations, sentences, logical form, and so on. Logical equivalence is one type of equivalence: it refers to the logical forms of the statements and asserts that their truth table columns are the same.

(c) $a-(-b)=a+b$.

(d) expressions.

(e) $(a \neq 0 \text{ and } ax+b=c) \iff x = \frac{c-b}{a}$.

(f) The true generalizations are: (i) (assuming $x \neq 0$ because it is not in the natural domain of $\frac{x^2+x}{x}$), (v), and (ix).

(g) Division bar, square-root bar, superscripts + subscripts

(h) NO ANSWER PROVIDED

Chapter 2:

(a) (i) $(2,5]$. (ii) $(-\infty,-1) \cup (2,\infty)$. (iii) $[-2,\infty)$

(b) $S \subset T$ iff $x \in S \Rightarrow x \in T$.

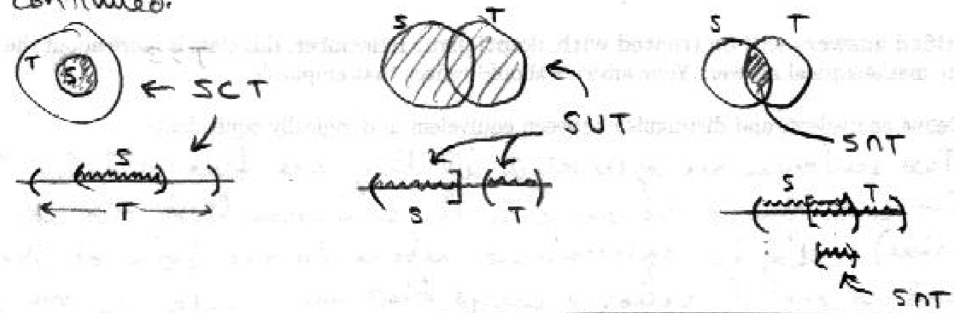
$x \in S \cup T$ iff $x \in S$ or $x \in T$.

$x \in S \cap T$ iff $x \in S$ and $x \in T$.

...continued...

Chapter 2, continued:

(b) continued.



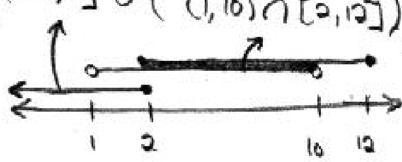
(c) $S=T$ iff $(x\in S \iff x\in T.)$

eg. $\{1,2,5,7\} = \{2,5,1,7\}$.

(d) $\left((1,2)\cup[3,5)\right)^c = (1,2)^c \cap [3,5)^c = \left((-\infty,1]\cup[2,\infty)\right)\cap\left((-\infty,3)\cup[5,\infty)\right)$

$= (-\infty,1]\cup[2,3)\cup(5,\infty)$

$\left((-\infty,2]\cup\left((1,10)\cap[2,12]\right)\right)^c = \left((-\infty,2]\cup[2,10)\right)^c$



$= \left((-\infty,10)\right)^c = [10,\infty)$

(e) (i) $f(y) = 2y-1$ (ii) $g(h+1) = 1-(h+1) = -h$.
(iii) $g(f(x)) = g(2x-1) = 1-(2x-1)$. (iv.) $f(f(x)) = f(2x-1) = 2(2x-1)-1$

(f) $f(x) = 2x-1$ is the same as $g(h(x))$, where $h(x) = 2x$, and $g(x) = x-1$.

(g) The rule on squaring (eg. $\sqrt{x+1} = x \Rightarrow x+1 = x^2$) can produce an extraneous solution. So can the rule on uniqueness of multiplication (eg. $\frac{(x+1)(x-1)}{(x+1)} = 2x \Rightarrow (x+1)(x-1) = 2x(x+1)$).

(h) The rule on canceling, if used incorrectly, can lead to dropped solutions. (eg. $\cancel{(x+1)}(x+2) = x\cancel{(x+1)} \Rightarrow x+2=x$ or $x+1=0$).

(i) $x = \sqrt{3x+10} \Rightarrow x^2 = 3x+10$ (on squaring)

$\Leftrightarrow x^2 - 3x - 10 = 0$ (u. add'n)

$\Leftrightarrow (x-5)(x+2) = 0$ (subst.)

$\Leftrightarrow x = 5$ or $x = -2$ (zero prod. rule)

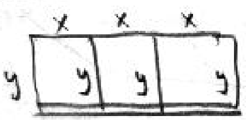
Check: $x = -2$ is not a solution, but $x = 5$ is ok.

(j) $x(x+1) = x(3x-2) \Leftrightarrow x+1 = 3x-2$ or $x = 0$ (rule on cancelling)

$\Leftrightarrow 3 = 2x$ or $x = 0$ (u. add'n)

$\Leftrightarrow x = \frac{3}{2}$ or $x = 0$ (u. mult. (by $\frac{1}{2}$)).

(k)



$A = xy$, and $3x + 4y = 100$.

Thus, $y = \frac{100-3x}{4}$, so $A = x\left(\frac{100-3x}{4}\right)$.

Chapter 3:

(a) NO ANSWER PROVIDED.

(b) A statement of the form $H \Rightarrow C$ is a conditional sentence.

(c) NO TRUTH TABLE PROVIDED, but you should find that it is not a tautology.

(d) NO TRUTH TABLE PROVIDED, but you should show that $A \Rightarrow B$ and $B \Rightarrow A$ have different truth values in some rows.

(e) Show that the truth table shows all T's (resp., all F's) in the last column.

(f) "$(x \geq 8 \Rightarrow x^2 \geq 64)$ and $(x \leq -8 \Rightarrow x^2 \geq 64)$."

(g) "If $S \subset T$ and $x \in S$, then $x \in T$."

(h) H and (not C). (Common mistake: $H \Rightarrow$ not C: this is incorrect!)

(i) not (A and B) is LE to not A or not B

not (A or B) is LE to not A and not B.

(j) $x+y > 2$ and $x \leq 1 \Rightarrow y > 1$.

Chapter 3, continued:

(K) NO ANSWERS PROVIDED.

(l) NO ANSWERS PROVIDED.

Chapter 4:

(a) $\forall$ is the universal quantifier (eg. "$\forall x,\ x^2 > 0$").
$\exists$ is the existential quantifier (eg. "$\exists x, y$ s.t $x > y$ and $x^2 < y^2$").

(b) A bound variable is quantified by either $\forall$ or $\exists$. A free variable is not quantified, so its values determine the truth of the sentence. Mathematicians use the words "placeholder" or "dummy variable" instead of "bound variable."

(c) • $2x + 3 = 7$ uses x as an unknown (the intent is to solve for x.)

• $A = \pi x^2$ uses x as a parameter that represents the radii of a whole family of circles. The intent is to express the formula for area of circles in general, not to solve for x.

(d) NO ANSWERS PROVIDED.

(e) "has some" is synonomous with "there exists", and existence statements are always interpreted as "one or more." Hence, even though there is only one 'b', it is correct to say "there are some b's."

(f) A true existence statement is usually easy to prove: just pick one example that works, and demonstrate that it really does work. To prove a generalization, you must usually prove it holds for an infinite number of cases.

(g) False generalizations can not be proven (they are false!), but a generalization can be proven to be false by finding a counterexample.

Chapter 4, continued:

(h) To disprove $\{1,2,3\} = [1,3]$, simply show that either $(\exists\ x \in \{1,2,3\}$ s.t. $x \notin [1,3])$ or $(\exists\ x \in [1,3]$ s.t. $x \notin \{1,2,3\})$

In this case, $x = 1.5$ suffices, for then $x \in [1,3]$ but $x \notin \{1,2,3\}$.

(i) $$y = \frac{-(x^2) \pm \sqrt{(x^2)^2 - 4(x)(2(x+1))}}{2(x)}$$

(j) i. "if $x < z$, then $f(x) < f(z)$."

ii. "f is not an increasing function iff $\exists\ x, z$ s.t. $x < z$ and $f(x) \geq f(z)$."

(K) i. 7 is an interior point of $[6,8]$, because $7 \in (6,8)$ and $(6,8) \subset [6,8]$.

ii. 7 is not an interior point of $\{7\}$, because no open intervals are contained in $\{7\}$.

iii. 3 is not an interior point of $[3,14)$ because any open interval that has 3 in it will also contain numbers that are less than 3, so it will not be contained in $[3,14)$.

iv. 4 is an interior point of $(3,14)$, because $4 \in (3.5, 4.5) \subset (3,14)$.

v. 1 is an interior point of $(0.9, 1.1)$ because $1 \in (.99, 1.01) \subset (0.9, 1.1)$.

(l) $$f(1,x) = f(2,-1) \iff 2(1) + 3(x) = 2(2) + 3(-1)$$
$$\iff 2 + 3x = 4 - 3$$
$$\iff 3x = -1 \iff x = -1$$

(m) i. $2 \cdot 5 \cos(5x) = 10 \cos(5x)$

ii. note: $a = 1$ and $b = \frac{1}{2}$. So the derivative is $\frac{1}{2}\cos\left(\frac{x}{2}\right)$.

(n) "A or B $\Rightarrow$ C".

on cases: "A $\Rightarrow$ C and B $\Rightarrow$ C", or "$x > 5 \Rightarrow |x| > 5$ and $x < -5 \Rightarrow |x| > 5$."

Form of negation: (A or B) and (not C) is the negation of the original.

(A and (not C)) or (B and (not C)) is the negation of the revised version. Note: these are L.E. forms!

Chapter 4, cont...

(o) "$\exists\, x, y$ s.t. $x^2 > y^2$ and $x \leq y$."

Chapter 5:

(a) $\left.\begin{aligned} A &\Rightarrow I_1 \\ &\Rightarrow I_2 \\ &\Rightarrow I_3 \\ &\Rightarrow B. \end{aligned}\right\}$ This proves $A \Rightarrow B$. That is, $S \subset T \Rightarrow T^c \subset S^c$.

(b) (i) Assume H is true, and work to deduce that C is true.

(ii) Assume that either A is true or that B is true (pick one, not both!) and work to deduce that the other one is equivalent using a chain of $\Leftrightarrow$ assertions.

(iii) Assume H and (not B) are true, and deduce that C is true.

(iv) To prove $(A \text{ or } B) \Rightarrow C$, you do two proofs: show $A \Rightarrow C$, and then show $B \Rightarrow C$.

(c) Proof: $x < y \Rightarrow 3x < 3y$ (T. 1.5.2)

$\Rightarrow 3x + 8 < 3y + 8$ (T. 1.5.1) QED.

(d) Proof: We need only show that "$\exists\, x$ s.t. $x^2 \leq x$" is true.

Let $x = \frac{1}{2}$. Then $x^2 = \frac{1}{4} \leq \frac{1}{2} = x$. QED.