Math 151 Worksheet - Section 4.2 Name:________________________________

Work (and argue) with your neighbor! Teaching and learning go hand-in-hand...

---

1. Consider the following sentences. Which variables are free, and which variables are placeholders? For placeholders, classify them by writing their (possibly implicit) quantifier. Also, when possible, classify free variables as unknowns or parameters / constants. (Some free variables cannot be classified further).

   (a) $2x + 1 = x$.

   (b) $S \subset [0, 1]$.

   (c) $x \in S \Longrightarrow x \in [0, 1]$.

   (d) $x + a < b$.

   (e) Let $f(x) = x - a$.

   (f) There exists S such that $S \cap T$ is nonempty.

   (g) There exists S such that $S \cap T$ is nonempty for all T.

---

2. Create a *single* generalization (about mulitiplication) to express the pattern among the following collection of facts. Use an appropriate quantifier and classifiy the variables you use as free or bound (placeholder).

   - $3(x + 2) = 3x + 6$
   - $-2(2 - x) = -4 + 2x$
   - $(x + 3)(3x + 1) = (x + 3)(3x) + (x + 3)(1)$

---

3. Consider the sentence

$$\text{For all x, } \mathrm{x}^2 + 4 = (\mathrm{x} + 2)^2$$

   (a) The sentence is quantified. What is the quantifier?

   (b) Give an example to show that the sentence is not true.

   (c) Given that the sentence is quantified, can we say the sentence is "ever true"? Explain.

   (d) Suppose the quantifier was changed to "There exists x such that..." Prove that this change makes the sentence true.

Answers: 1(a) x is free (unknown). 1(b) S is free. 1(c) x is quantified ("for all x"), S is free. 1(d) x is free (unknown); a and b are free (parameters or constants). 1(e) x is a placeholder ("for all x"); a is free (parameter). 1(f) S is a placeholder ("there exists S"); T is free. 1(g) S is still a placeholder, but now so is T ("for all T").

2) For all a, b, and c, $a(b + c) = ab + ac$. All three variables are placeholders.

3(a) "For all x..." 3(b) $x = 1$. 3(c) No. The universal quantifier (for all) requires that we simultaneously consider all possible values for x and see that at least one of them makes the sentence false, and thus we must conclude that the sentence is simply false. 3(d) To prove that "there exists" x that makes the sentence true, we just have to find one: $x = 0$ works in this case.

4. The string of letters *aaaaa* satisfies the condition "All the letters are a's." Give a different string of five letters to satisfy each of the following conditions:

   (a) "All the letters are not a's."

   (b) "Not all the letters are a's."

   (c) "Some of the letters are not a's.

5. Which of the phrases, "All are not a's," "Some are not a's," and "Not all are a's," are applicable to the following strings of letters? (There may be more than one!)

   (a) *aabba*

   (b) *bcbcb*

6. Restate each of the following sentences using abbreviated mathematical notation. Use abbreviations like $\forall$, $\exists$, and s.t., $\Longrightarrow$, and so on, where applicable.

   (a) "For all real numbers x, $x^2 \geq 0$."

   (b) "Suppose there exists S such that $S \cap T$ is the empty set."

   (c) "If $x < 0$, then $|x| = -x$."

   (d) "For any sets S and T, S is a subset of $S \cup T$."

   (e) "For all real numbers y, there exists x such that $y = 2x$."

4(a) *bcbbd*. 4(b) *aaaab*. 4(c) *aaaab*. 5(a) "Some are not," "Not all are." 5(b) "All are not," "Some are not," and "Not all are." 6(a) "$\forall x \in \mathbb{R}$, $x^2 \geq 0$." 6(b) "Suppose $\exists S$ s.t. $S \cap T = \phi$." 6(c) "$x < 0 \Longrightarrow |x| = -x$." 6(d) $\forall\, S$ and T, $S \subset (S \cap T)$." 6(e) "$\forall\, y, \exists\, x$ s.t. $y = 2x$."