**M182 Lecture Worksheet, Sec. 8.3** **Name:**______________________________

**Applications, Part II: Moments and Centers of Mass**.

**Definition:** Archimedes knew that if two masses $m_1$ and $m_2$ are placed on a "teeter-totter", and they are located at $d_1$ and $d_2$ units from the pivot point, then the teeter-totter will balance when

$$m_1 d_1 = m_2 d_2$$

The number $(m_1 d_1)$ is called the **moment** of the mass $m_1$. So your teeter-totter will balance precisely when the moments of the two masses are equal.

In two dimensions, we define $M_x$ to be the moment about the $x$-axis and $M_y$ the moment about the $y$-axis, and we treat these moments independently. The **center of mass**, or **centroid**, of a two-dimensional system is defined as the point $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m}$$

with $m$ representing the total mass of the system. Physically, the center of mass is an important concept; any moving object behaves as if its entire mass was centered at this key point.

It is easy to find the center of mass of a discrete system; however, the interesting problems require calculus. First, let us examine an "easy" two dimensional discrete system.

---

1. **Example:** Find the moments and the center of mass of the two-dimensional system with masses 2, 4, 5, and 10 located at coordinates $(1, 1)$, $(-2, 3)$, $(-2, -4)$, and $(2, 5)$, respectively.

**Finding the centroid of a flat plate with uniform density**.We can use the principles we use for discrete systems to develop a method for finding the centroid of an arbitrary thin plate. The trick is to divide the region into a discrete collection of thin rectangles and then use integration to sum up the moments.

(a) First, sketch the given region and a typical approximating rectangle with height $f(x)$ and width $dx$ (Riemann Sum style).

(b) Use symmetry! If the region is symmetric about a line, at least one coordinate of the centroid may be obvious!

(c) By symmetry, the centroid of a rectangle lies at its geometric center. Label the centroid of the rectangle you drew, and also label the points at the top and bottom of the rectangle.

(d) Use your picture to find the moments about the $x$- and $y$-axes. Remember, $M_x = my$ and $M_y = mx$, and the mass of the rectangle will be $(\rho \cdot \text{length} \cdot \text{width})^*$.

(e) Integrate to find the moments $M_x$ and $M_y$ of the entire region.

(f) Find the total mass $m$ of the system via integration, and then multiply by the density $\rho^*$.

(g) The centroid is $(\bar{x}, \bar{y})$, where $\bar{x} = \dfrac{M_y}{m}$ and $\bar{y} = \dfrac{M_x}{m}$

*You'll find that the constant density $\rho$ will always cancel out once you divide by the total mass of the system, so you may ignore it when calculating the center of mass. (Don't ignore it if you are asked for the moments $M_x$ and $M_y$, however!)

---

2. **Example:** Find a formula to compute the moments $M_x$ and $M_y$ of the region under the curve $y = f(x)$ between $x = a$ and $x = b$.

3. **Example:** Find the center of mass of a semicircular plate of radius $r$.

4. **Example:** Calculate the center of mass of the region between the graphs of $f$ and $g$ on the interval $[1, 2]$, where

$$f(x) = (x + 1)^2 \text{ and } g(x) = (x - 1)^2$$

---

5. **Assigned Homework**: pg. 562-563, numbers 19, 20, 23, 27, *32, *39