M182 Lecture Worksheet, Sec. 11.1 **Name:**______________________________

Sequences (Continued). We now list several important theorems and definitions that we will need as we continue our study of sequences. You should compare these with the analogous theorems and definitions for real-valued functions.

- **Squeeze Theorem:** If $a_n \leq b_n \leq c_n$ for all $n > n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n$ is also equal to L.
- **Definition:** A sequence is called **increasing** if $a_{n+1} \geq a_n$. A sequence is called **decreasing** if $a_{n+1} \leq a_n$ for all $n \geq 1$.
- **Definition:** A sequence is called **monotone** (or **monotonic**) if it is either increasing or decreasing.
- **Definition:** A sequence is said to have a property **eventually** if it has that property after finitely many terms have been removed.
- **Definition:** A sequence is **bounded above** if there is a number M such that $a_n \leq M$ for all $n \geq 1$. **Bounded below** is defined similarly.
- **Definition:** A sequence is said to be **bounded** if it is both bounded above and bounded below.
- **Monotone Convergence Theorem**: All bounded monotone sequences converge.

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1. **Example:** Prove that the sequence $\{n - 2^n\}_{n=1}^{\infty}$ is monotonic.

2. **Example:** Show that $\left\{\frac{n!}{5^n}\right\}_{n=1}^{\infty}$ is eventually increasing. (*Hint: Consider* $\frac{a_{n+1}}{a_n}$)

3. **Example:** Use a calculator to graph the sequence $a_n = \dfrac{n}{n^2+1}$. Then prove that it is decreasing.

4. **Example:** Use a calculator to graph the sequence $\left\{ \dfrac{\sin n}{\sqrt{n}} \right\}$. Then prove that this sequence converges to zero.

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5. **Example:** Consider the recursive sequence defined by $a_{n+1} = \dfrac{2}{3}\left(a_n + \dfrac{1}{a_n^2}\right)$.

   (a) Let $a_0 = 2$, and use a calculator to compute the next four terms a_1, a_2, a_3, a_4 of the sequence.

   (b) Show that $a_{n+1} \le a_n$ for all $n \ge 0$. (Hint: show that $a_{n+1} - a_n \le 0$.)

   (c) Give a lower bound for $\{a_n\}$.

   (d) Since $\{a_n\}$ is decreasing and bounded below, the Monotone Convergence Theorem says the sequence converges. What is $\lim\limits_{n\to\infty} a_n$?

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6. **Homework**: pg. 702, numbers *50, 55, 57, *59, *63