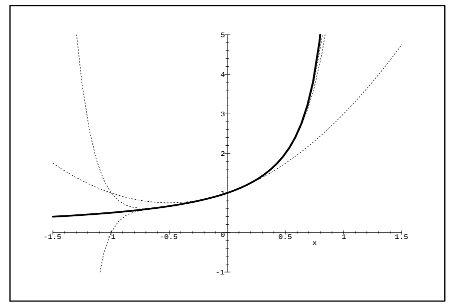
M182 Lecture Worksheet, Sec. 11.9 **Name:**________________________________

Representing Functions as Power Series. We have seen that for $|x| < 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + ... = \sum_{k=0}^{\infty} x^k$$

We have represented the function $f(x) = \frac{1}{1-x}$ with a power series. Notice that the partial sums s_n for this series are just a bunch of polynomials. For example:

$s_2 = 1 + x + x^2$
$s_8 = 1 + x + x^2 + x^3 + ... + x^7 + x^8$
$s_{11} = 1 + x + x^2 + x^3 + ... + x^9 + x^{10} + x^{11}$



We have plotted these three polynomials (dashed lines) along with the graph of $f(x) = \frac{1}{1-x}$ (solid line). As $n \to \infty$, these polynomials do a better and better job of approximating the function $f(x)$ between -1 and 1.

Notice the difference between the expressions below. The first is an **approximation** to the function $f(x) = \frac{1}{1-x}$ by a fourth degree polynomial. The second is a statement about equality; the function f and the power series are **exactly equivalent** for $|x| < 1$.

$$\frac{1}{1-x} \approx 1 + x + x^2 + x^3 + x^4$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + ... \text{ for } |x| < 1.$$

It is a major result in advanced calculus that power series can be integrated and differentiated *term-by-term*, just as finite polynomials can be! This powerful theorem allows us to calculate integrals (and derivatives) of functions such as $f(x) = \sin\left(\frac{\pi}{2}x^2\right)$ and $g(x) = e^{x^2}$, neither of which have elementary anti-derivatives.

Theorem (Term-by-term Integration and Differentiation of Power Series)

If the power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ has radius of convergence $R > 0$, then the function

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + ... = \sum_{k=0}^{\infty} c_k(x-a)^k$$

is differentiable (and hence continuous) on $(a - R, a + R)$, and the series

$$f'(x) = \frac{d}{dx}\left[\sum_{k=0}^{\infty} c_k(x-a)^k\right] = \sum_{k=0}^{\infty} \frac{d}{dx}\left[c_k(x-a)^k\right]$$

$$\int f(x)\,dx = \int \left[\sum_{k=0}^{\infty} c_k(x-a)^k\right] dx = \sum_{k=0}^{\infty} \left[\int c_k(x-a)^k\,dx\right]$$

both have radius of convergence R.

Note 1: The theorem is meant to be applied only to power series; it need not hold for other kinds of series.

Note 2: Even though the radius of convergence remains the same, the *endpoints* of the interval of convergence need to be rechecked.

1. **Example:** Find a power series representation of $f(x) = \dfrac{1}{1+x^2}$, and state the interval of convergence.

2. **Example:** Find a power series representation of $f(x) = \dfrac{x}{1+x}$.

3. **Example:** Use term-by-term integration and the fact that $\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1} x$ to find a power series representation for $\tan^{-1} x$, and state its interval of convergence.

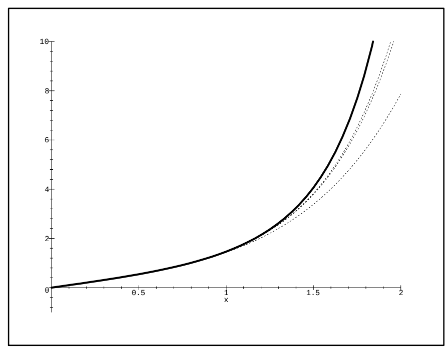
4. **Example:** The function $y = e^{x^2}$ has no elementary antiderivative.

   (a) Use term-by-term differentiation to verify the fact that the power series representation for e^x is
   $$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

   (b) Use the power series for e^x to calculate a power series for $\displaystyle\int e^{x^2}\,dx$.

   (c) Calculate the polynomials s_3, s_4, and s_5:

   (d) Using Maple, I have used a numerical technique to plot $\int_0^x e^{t^2}\,dx$ for x between 0 and 2 (solid line). I have also included the graphs of s_3, s_4, and s_5 as dashed lines. Notice that they do a fair job of approximating the solution near $x = 0$, but the approximation worsens as we move away from the origin.



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5. **Homework**: pg. 749-750, numbers 1, 2, 3, 5, 7, 11, 25, 37.