M182 Lecture Worksheet, Sec. 11.10 **Name:**______________________________

Taylor and MacLauren Series. In previous sections, we have seen examples of how certain functions can be represented as power series. Here, we learn how to find the power series representation for a wide variety of functions.

The first few examples below are designed to illustrate how the general **Taylor Series** can be derived. For your reference, the conclusions are summarized below.

Theorem (Taylor Series Representations): If f has a power series representation (expansion) at a, then it must be of the form

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots .$$

The formula for the Taylor Series is simplified when $a = 0$. This special case (when the Taylor Series is centered at the origin) is called the **MacLauren Series**, and the above formula simplifies to

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots .$$

It is important to note that **there do exist functions that are not equal to their power series.** An innocent-looking example is $f(x) = e^{(-1/x^2)}$. In this course, we will restrict ourselves to those functions that do have a power series expansion.

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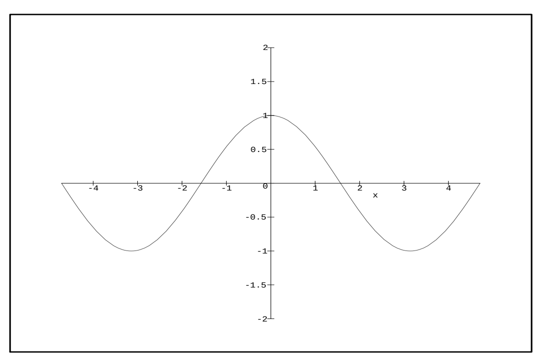
1. **Motivating Example:** Develop a model for the function $f(x) = \cos x$ near $x = 0$ by using a formula of the form $T_e(x) = ae^x + bx + c$. Sketch the graph of T_e on top of the graph of $\cos x$ given below. (*Hint: The idea is to make sure that* $T_e(0) = f(0)$, *and* $T_e'(0) = f'(0)$, *etc.*)



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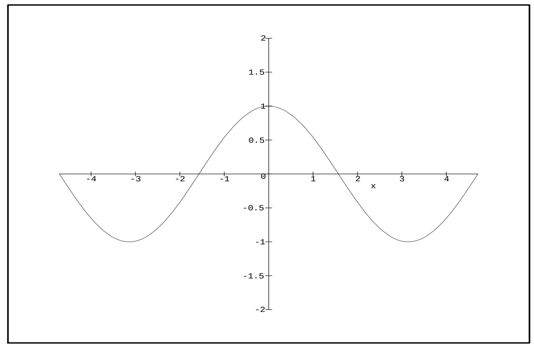
2. **A better model:** The previous model is somewhat inconvenient because it cannot be extended to improve the accuracy of our approximation. Polynomials are much simpler, and they can be made more accurate just by including higher powers of x.

Model the function $f(x) = \cos x$ near $x = 0$ by using a parabola $T_2(x) = a + bx + cx^2$, and sketch the graph of $T_2(x)$ on top of the graph of $\cos x$ given below.

(**Notation:** $T_2(x)$ is called the Taylor polynomial of degree 2.)



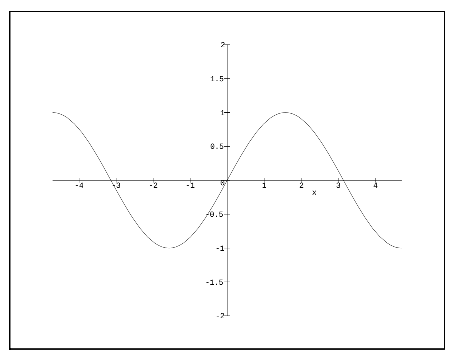
3. **Example:** On the previous lecture worksheet, we derived a power series representation for e^x by solving a differential equation. Use the Taylor Series Theorem to find the MacLauren Series for e^x, and verify that it matches the formula you obtained in section 11.9.

4. **Example:** Derive the full MacLauren Series for $f(x) = \cos x$, and find its radius of convergence. (Note: you may assume that $\cos x$ can be expressed as a power series.)

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0		
1		

5. **Extension:** Use the MacLauren Series for $\cos x$ to find the power series expansion of $\sin x$ by integrating term-by-term.

6. **Another Extension:** Use a calculator to plot $\sin x$ along with $T_7(x)$, the 7th degree Taylor polynomial centered at $x = 0$.



7. **Example:** Find the power series for (a) $x^2 \sin x$ and (b) $\sin(x^2)$.

8. **Example:** Estimate $\int_0^1 e^{-x^2}\,dx$ by using the power series for e^x and summing the first 5 terms. Be sure to get a bound on your error!

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9. **Homework**: pg. 760-761, numbers 3-5, 9, 13, 15, 21, 23, 25, 27, 29, 33, 37, 38 (see 29)