Two Simple Examples in Non-Euclidean Geometry

by

Shing-Seung So

Zuwaina S. Al-Maskari

Introduction. The idea of taxicab geometry comes from the every day experience of driving a car from one location to another location in the city. Mathematically, we may consider the two locations as points $A(a_1, a_2)$ and $B(b_1, b_2)$. Certainly, when we go to location B from A, we must stay on the streets and cannot cut diagonally across any block. Thus Euclidean distance,

$$d_E(A,B)=\sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2},$$

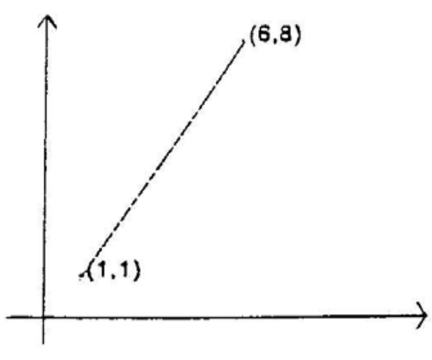
cannot be used in solving this problem. On the contrary, the taxicab distance,

$$d_T = |b_1 - a_1| + |b_2 - a_2|,$$

can realistically reflect the actual distance travelled.

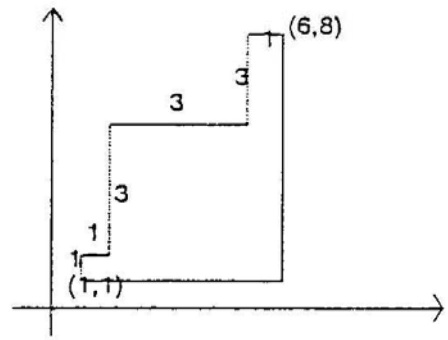
As an example, let A(1, 1) and B(6, 8) be two points. Then $d_E(A,B) = \sqrt{74}$ and $d_T(A, B) = |6 - 1| + |8 - 1| = 12$. Referring

Figure 1



to Figures 1 and 2, one can see that Figure 2 represents the actual distance travelled.

Figure 2



Although taxicab geometry is almost the same as Euclidean coordinate geometry, it is shown in this paper by two simple examples that the former is in fact classified as non-Euclidean geometry, which is a geometry whose marked characteristic is its rejection of Euclid's parallel postulate as stated in [1].

Although Euclid's postulate is well known to geometry teachers, for completeness sake we would still include it as statement (1) in the following equivalent statements found in Euclidean geometry. Interested readers may refer to [2] for additional information.

(1) If a transversal cuts two lines so that the angles formed on one side of the transversal is less than two right angles, then the two lines will meet on that same side.

(2) Through a point not on a given line

Shing-Seung So, Ph.D. and Zuwaina S. Al-Maskari, M.S.; Department of Mathematics and Computer Science; Central Missouri State University; Warrensburg, MO 64043.

there passes not more than one parallel to the line.

(3) Two lines that are parallel to the same line are parallel to each other.

(4) A line that meets one of two parallels also meets the other.

(5) Parallel lines are equidistant from one another.

Main Results. First, we are going to examine the definition of a line in Euclidean geometry and taxicab geometry.

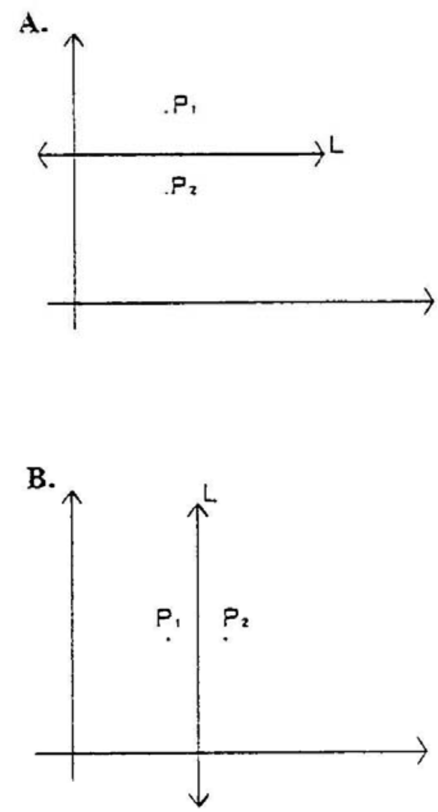
In [3], Iny pointed out that the most natural way of defining a line in geometry is to consider it as the locus (set of points) which is equidistant from two given points, say P_1 and P_2 . For instance, a line in Euclidean geometry can be viewed as the set of points which form the perpendicular bisector of two given points.

Using the same idea, we define *a line in taxicab geometry* (or a taxicab bisector) as follows. If P_1 and P_2 are two distinct points in the plane and d_T is the taxicab distance function, then the set $L = \{P(x,y) | d_T(P, P_1) = d_T(P, P_2)\}$ is called the *taxicab bisector* of the line segment P_1P_2 or the taxicab bisector of the points P_1 and P_2.

Unlike lines in Euclidean coordinate geometry, which are characterized by slopes, taxicab bisectors (or lines in taxicab geometry) can be completely characterized into the following four forms. The description of their constructions will be given. Interested readers can refer to [4] for more details.

1. If P_1 and P_2 have either the same x-coordinates or the same y-coordinate, then the taxicab bisector of P_1 and P_2 is the same as the one in Euclidean geometry (see Figures 3).

Figure 3



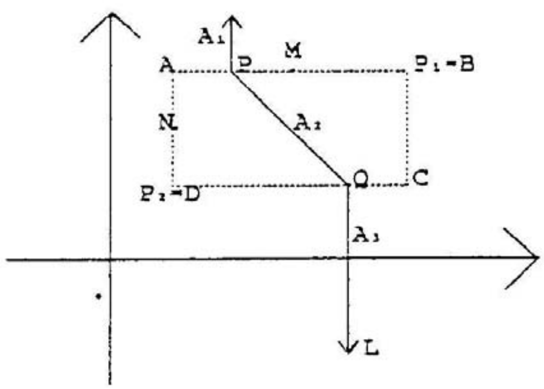
2. If P_1 and P_2 have different x and y coordinates, then we construct the taxicab bisectors as follows.

Step 1. Draw vertical and horizontal lines through P_1 and P_2 to form a rectangle ABCD so that they are the opposite corners of a rectangle. For the sake of convenience, A is always the upper left-hand corner and C is the lower right-hand corner.

Step 2. (a) In the rectangle ABCD, if $|AB| > |AD|$, $P_1 = B$, and $P_2 = D$, then first locate the midpoints M and N of $\overline{AB}$ and $\overline{AD}$, respectively. Mark the point P which is $|DN|$ units to the left of M. Label Q as the point of intersection of the line through P with slope -1 and the line segment CD. Let A_1 be the vertical ray above $\overline{AB}$ having P as its end point, let A_2 be the line segment PQ, and let A_3 be the vertical ray below $\overline{DC}$ having Q as its end point. Then L, which is the union of the sets A_1, A_2, and A_3, is the taxicab bisector of the points P_1 and P_2 (see Figure 4).

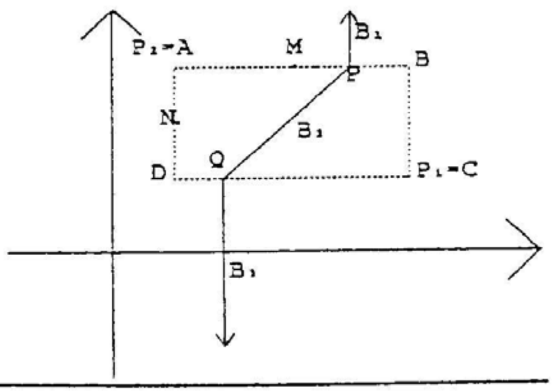
(b) Similarly, if $|AB| > |AD|$, $P_1 = C$, and $P_2 = A$, then we can construct B_1, B_2, and B_3 as in Part (a). Then L, which is

Figure 4



the union of the sets B_1, B_2, and B_3, is the taxicab bisector of the points P_1 and P_2 (see Figure 5).

Figure 5



(c) If $|AB| < |AD|$, $P_1 = B$, and $P_2 = D$, then we can obtain the graph of the taxicab bisector of P_1 and P_2 by rotating the Figure 5 by 90° in the clockwise direction (see Figure 6A).

(d) In the rectangle ABCD, if $|AB| < |AD|$, $P_1 = C$, and $P_2 = A$, then we can obtain the graph of the taxicab bisector of P_1 and P_2 by rotating Figure 4 by 90° in the clockwise direction (see Figure 6B).

(e) If $|AB| = |AD|$, $P_1 = B$, and $P_2 = D$, then first obtain the region F_1 by shading the part which is on or to the left of $\overline{AP_2}$ and on or above $\overline{AP_1}$. Similarly, the region F_3 can be obtained by shading the region which is on or to the right of $\overline{P_1C}$ and on or below $\overline{P_2C}$. Finally, F_2 is the diagonal AC of the square ABCD. Then the taxicab bisector of P_1 and P_2 is the union of the sets F_1, F_2, and F_3 (see Figure 7).

Figure 6

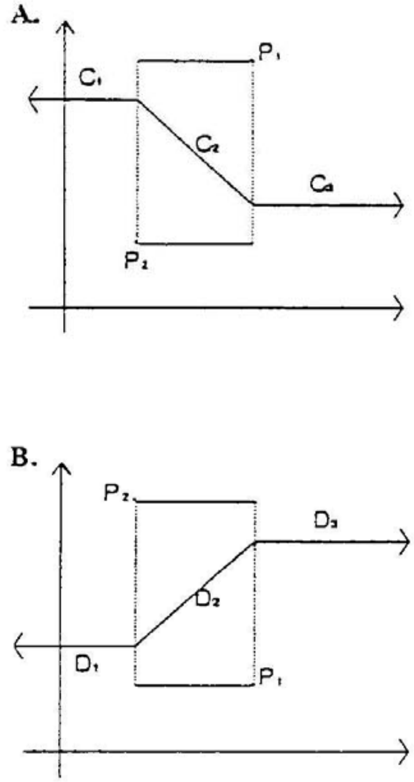
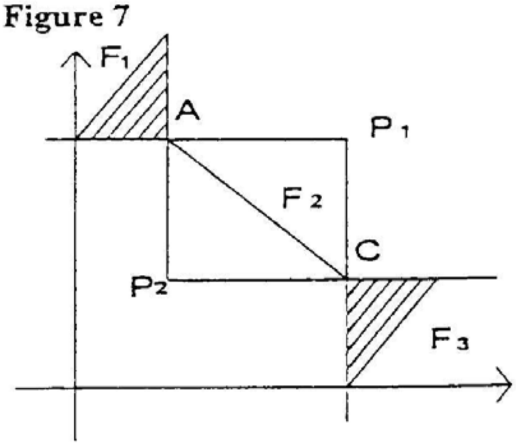
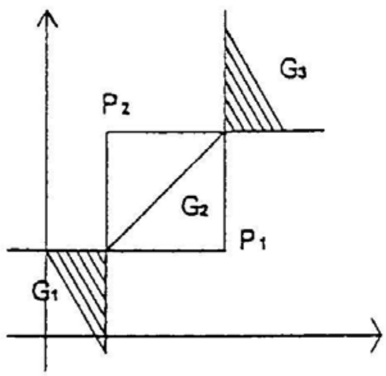


Figure 7



(f) Similarly, if $|AB| = |AD|$, $P_1 = C$, and $P_2 = A$, then the taxicab bisector of P_1 and P_2 is the union of the sets G_1, G_2, and G_3 (see Figure 8).

Figure 8



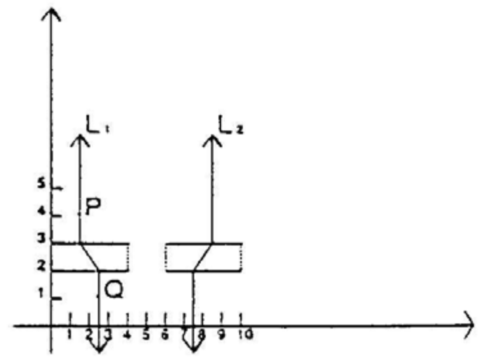
Next, we would like to point out how one can find the shortest taxicab distance from a point $P(x_1, y_1)$ to a line of the form $Ax + By + C = 0$. If the absolute value of the slope of the line is greater than 1, then the horizontal distance from P to the line will be the shortest taxicab distance. If the absolute value of the slope of the line is less than 1, then the vertical distance from P to the line will be the shortest taxicab distance. If the slope is 1 or -1, then either the horizontal or the vertical distance from P to the line will do.

Remark. Two taxicab bisectors L_1 and L_2 are said to be *parallel* if L_1 and L_2 do not intersect..

Example 1. Suppose L_1 is the taxicab bisector of the points $P_1(4, 3)$ and $P_2(0, 2)$ and L_2 is the taxicab bisector of the points $Q_1(10, 2)$ and $Q_2(6, 3)$.

Let P(1.5, 4) and Q(2.5, 1) be two points on L_1. Then L_1 and L_2 do not intersect, but the distance from P to L_2 is 7 and the distance from Q to L_2 is 5. Therefore, L_1 and L_2 are not equidistant (see Figure 9).

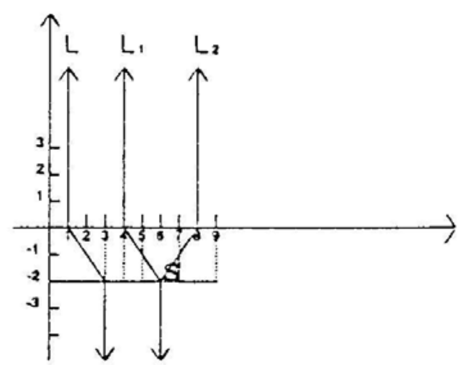
Figure 9



Example 2. Suppose L is the taxicab bisector of the points $P_1(4, 0)$ and $P_2(0, -2)$, L_1 is the taxicab bisector of the points R_1 (7, 0) and $R_2(3, -2)$, and L_2 is the taxicab bisector of the points $Q_1(5, 0)$ and $Q_2(9, -2)$.

Consider S(6,-2). Then L and L_1 do not intersect, L and L_2 do not intersect, S is on both L_1 and L_2, and $L_1 \neq L_2$ (see Figure 10).

Figure 10



Remark. Example 2 indicates that the equivalent statements (2) through (4) in Euclidean geometry, which were mentioned in the introduction are invalid in taxicab geometry, and hence taxicab geometry is non-Euclidean.

REFERENCES

1. G. James, R. James, Mathematics Dictionary, D. Van Nostrand Company, Inc. New York, 1960.
2. D. Gans, An Introduction to Non-Euclidean Geometry, Academic Press, New York, 1973.
3. D. Iny, Taxicab Geometry: Another Look at Conic Sections, *Pi Mu Epsilon Journal*, Vol. 7, No. 10 (1984), 645-647.
4. Y. Liu, Conic Sections in Taxicab Geometry, preprint, CMSU.
5. J. Moser and J. Kramer, Lines and Parabolas in Taxicab Geometry, *Pi Mu Epsilon Journal*, Vol. 7, No. 7 (1982), 441-444.