9.3 One Analytic Technique – Separation of Variables

Standard 1^{st} order D.E. $\frac{dy}{dx} = f(x,y)$

Example ① $\frac{dy}{dx} = 15x^2 - 4$

$y(x) = \int (15x^2 - 4)\,dx$ ← looks like $\int dy = \int (15x^2-4)\,dx$

$\boxed{y(x) = 5x^3 - 4x + C}$ ← general solution

② $\frac{dy}{dx} = 15x^2 - 4$ $\quad y(0) = 4$ ← initial condition

⋮

$y(x) = 5x^3 - 4x + C$

$4 = y(0) = C$

$\therefore \boxed{y(x) = 5x^3 - 4x + 4}$

Separable Equations $\quad \frac{dy}{dx} = g(x)\,h(y)$

"separate parts"

If $h(y) \neq 0$ $\quad \frac{1}{h(y)}\frac{dy}{dx} = g(x)$

$$\int \frac{1}{h[y(x)]}\frac{dy}{dx}\,dx = \int g(x)\,dx$$

let $u = y(x)$

$du = \frac{dy}{dx}\,dx$

$$\int \frac{1}{h(u)}\,du = \int g(x)\,dx$$

let $y = u$

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

"Looks like... separate y and t parts... integrate"

Examples

① $\dfrac{dy}{dx} = xy^2$

If $y \neq 0$ $\quad \displaystyle\int \frac{1}{y^2}\,dy = \int x\,dx$ ← need integral right away to be correct

$$-\frac{1}{y} + C_1 = \frac{x^2}{2} + C_2 \qquad C_1, C_2 \in \mathbb{R}$$

$$-\frac{1}{y} = \frac{x^2}{2} + C_3 = \frac{x^2 + 2C_3}{2} \qquad \text{let } C_3 = C_2 - C_1$$

$$\text{Let } C_4 = 2C_3$$

$$-\frac{1}{y} = \frac{x^2 + C_4}{2}$$

$$\boxed{y = \frac{-2}{x^2 + C_4} \qquad C_4 \in \mathbb{R}}$$

Check $y=0$ → $y=0$ is also a solution (equilibrium)

② $\dfrac{dy}{dx} = 2(y-1)x$

If $y \neq 1$ $\quad \displaystyle\int \frac{dy}{y-1} = \int 2x\,dx$

$$\ln|y-1| = x^2 + C_1 \qquad C_1 \in \mathbb{R}$$

$$|y-1| = e^{x^2+C_1} = e^{C_1}e^{x_2} = C_2 e^{x_2}, \qquad C_2 = e^{C_1} > 0$$

$$y - 1 = \pm C_2 e^{x^2}$$

$$\begin{cases} y = C_3 e^{x^2} + 1 & C_3 = \pm C_2 \neq 0 \\ y = 1 \text{ also works in D.E.} \end{cases}$$

Check $y=1$

$$\boxed{y(x) = C_4 e^{x^2} + 1, \quad C_4 \in \mathbb{R}}$$

③ $\frac{dy}{dx} = 2(y-1)x \qquad y(0) = 3$

$\vdots$

$$y(x) = C_4 e^{x^2} + 1$$

$$3 = y(0) = C_4 + 1 \Rightarrow C_4 = 2$$

$$\boxed{y(x) = 2e^{x^2} + 1}$$

③ Unlimited Growth

$$\frac{dP}{dt} = kP$$

If $P \neq 0$ $\qquad \int \frac{dP}{P} = \int k\,dt$

$$\ln|P| = kt + C_1 \qquad C_1 \in \mathbb{R}$$

$$|P| = e^{kt+C_1} = e^{C_1}e^{kt} = C_2 e^{kt}, \quad C_2 = e^{C_1} > 0$$

Check $P=0$ $\qquad \begin{cases} P = \pm C_2 e^{kt} = C_3 e^{kt}, & C_3 = \pm C_2 \neq 0 \\ P = 0 \text{ ok too} \end{cases}$

$$\boxed{P(t) = C_4 e^{kt}, \quad C_4 \in \mathbb{R}}$$

Homework 9.3 # 1–17 (every other odd) 19, 35, 39, $\underbrace{41, 43}$	Also solve Logistic Problem with $k=1$ and $K=4$ $\frac{dP}{dt} = (1-\frac{P}{4})P$

↑ go over

NE

First Order

Linear Differential Equations 9.5

In this section will assume t is positive (time)

(In class use x instead of t)

$$\frac{dy}{dt} = g(t)\,y + r(t)$$

If $r(t) = 0 \rightarrow$ easy way

$$\frac{dy}{dt} = g(t)\,y \qquad \text{separable!}$$

(skip)

Separable

$$\int \frac{dy}{y} = \int g(t)\,dt \qquad (\text{if } y \neq 0)$$

$$\ln|y| + C_1 = \int g(t)\,dt$$

$$\ln|y| = \int g(t)\,dt - C_1$$

$$|y| = e^{-C_1} e^{\int g(t)\,dt}$$

$$y = \underbrace{\pm e^{-C_1}}_{\text{non zero}} e^{\int g(t)\,dt}$$

(zero equiv. sol.)

$$y = Ae^{\int g(t)\,dt}, \quad A \in R$$

If $r(t) \neq 0$

(1) $$\frac{dy}{dt} = g(t)\, y + r(t)$$

$$\frac{dy}{dt} - g(t)\, y = r(t)$$

Easier notation, let $a(t) = -g(t)$

(2) $$\underbrace{\frac{dy}{dt} + a(t)\, y}_{} = r(t)$$

What do you take
the derivative of to
get this? ... looks like
a product rule ... some fn of t times y
try $e^{\int a(t)\,dt}$

(3) Note $$\frac{d}{dt}\left(e^{\int a(t)\,dt}\, y\right) = e^{\int a(t)\,dt}\, y' + e^{\int a(t)\,dt}\, a(t)\, y$$

$$= e^{\int a(t)\,dt}\left(y' + a(t)\, y\right)$$

$$= e^{\int a(t)\,dt}\, r(t)$$

(4) $$\therefore\ e^{\int a(t)\,dt}\, y = \int e^{\int a(t)\,dt}\, r(t)\, dt$$

$$y = e^{-\int a(t)\,dt} \int e^{\int a(t)\,dt}\, r(t)\, dt$$

Examples

1) $\frac{dy}{dt} = -2y + 5$ Find general solution

$*$ $\frac{dy}{dt} + \underline{\underline{2}}y = 5$ $*$

can use case where arb const is zero

$$\frac{d}{dt}\left(e^{\int 2dt} y\right) = \underline{\frac{d}{dt}\left(e^{2t} y\right)}$$

$$= e^{2t} y' + e^{2t} 2y$$

$$= e^{2t}(y' + 2y)$$

$$\overset{*}{=} \underline{e^{2t} 5}$$

Integrate

$$e^{2t} y = \int e^{2t} 5\, dt$$

$$= \frac{5}{2} e^{2t} + c$$

$$y = e^{-2t}\left(\frac{5}{2}e^{2t} + c\right)$$

$$\boxed{y = \frac{5}{2} + ce^{-2t}}$$

Check: $y' = -2ce^{-2t}$

$$-2y + 5 = -5 - 2ce^{-2t} + 5 = -2ce^{-2t}$$

2) $\frac{dy}{dt} = \frac{6}{t} y + t^3$, $y(1) = 1$ Specific Solution

* $y' - \underline{\underline{\frac{6}{t}}} y = t^3$

$$\frac{d}{dt}\left(e^{-\int \frac{6}{t} dt} y\right) = \frac{d}{dt}\left(e^{-6\ln(t)} y\right)$$

$$= \frac{d}{dt}\left(e^{\ln t^{-6}} y\right)$$

$$= \underline{\frac{d}{dt}\left(t^{-6} y\right)}$$

$$= t^{-6} y' + -6t^{-7} y$$

$$= t^{-6}\left(y' - \frac{6}{t} y\right)$$

$$\overset{*}{=} t^{-6}\, t^3$$

$$= \underline{\underline{t^{-3}}}$$

Integrate ...

$$t^{-6} y = \int t^{-3}\, dt$$

$$= -\frac{1}{2} t^{-2} + C$$

$$y = t^{+6}\left(-\frac{1}{2} t^{-2} + C\right)$$

$$-\frac{1}{2} t^4 + C t^6$$

$$1 = y(1) = -\frac{1}{2} + C \quad \Rightarrow C = 3/2$$

$$\boxed{y(t) = -\frac{1}{2} t^4 + 3/2\, t^6}$$

HW. § 9.5 # 1-19 (odd)
27, 33, 35