to my parents

Acknowlegments

It is with pleasure that I thank the many people who have supported me in this work.

First, I would like to thank my advisors, Professor Richard Rochberg and Professor Mitchell Taibleson. Their guidance and encouragement have made this work possible and also enjoyable. From them, I have learned both mathematics and teacher dedication.

I would also like to thank Professor Michael Frazier for many helpful discussions. Members of my committee have also helped with their time and assistance.

My years in St. Louis have been a time of growth and of happiness. I would like to thank the many students, staff and faculty at Washington University whose friendships have made my time in graduate school very special. These friendships are what I treasure most from my time in St. Louis. I especially would like to thank Dr. Barbara Weiss and Dr. Guido Weiss for all they have done for me in this past year, and I would like to thank Dr. Estela Gavosto and Dr. Rodolfo Torres for their close friendship and support.

Finally, I would like to thank my Mom, Dad, brothers Mike and Dave, and sister Erin for their love and encouragement.

Contents

List of Tables

Chapter 0

Introduction

A wavelet basis is an orthonormal basis of smooth functions generated by dilations by 2^{-m} and translations by $n2^{-m}$ of a single function. While the discovery of such bases has only occurred in the last ten years or so, earlier results in Harmonic Analysis have provided the ground work for this new topic. Wavelet theory is a culmination of many ideas in this area. A brief history of some of the ideas for wavelets will be given in Chapter 1.

Wavelet theory has caught the interest of many mathematicians as well as engineers and other scientists who are interested in its many applications. Applications for wavelets include uses in medicine for faster and sharper scanners for medical diagnosis, more efficient computer storage techniques for large amounts of data, and sharper audio and visual signals with less information needed. Wavelets are beneficial in these applications because of the highly efficient ways in which a wavelet series can represent a signal. This allows for better reproductions of functions with less data needed. More details of this will be given in the chapters to follow.

In this paper, we will look at pointwise convergence properties of various types of wavelets. Because the wavelet bases are made up of dilations of a function, the wavelet functions are able to zoom in on any part of a function being evaluated.

In Chapter 1, background for a motivation for wavelets will be given along with examples of various types of wavelet bases which will be explored in more detail in following chapters. The underlying structure, which has been hinted at in this Introduction, will be discussed in greater detail. Also, a comparison between a local property of wavelet expansions compared to expansions by the standard and a windowed Fourier transform will be given.

In Chapter 2, pointwise convergence properties will be found for wavelets in $\mathcal{S}(\mathbf{R})$, that is wavelets with rapid decrease. The convergence for the wavelets in $\mathcal{S}(\mathbf{R})$ of P.G. Lemarié and Y. Meyer [Le-Me] will specifically be examined. In particular, convergence of a wavelet expansion at a Lebesgue point will be proved. Rates of convergence for wavelet expansion of a function at points with specific smoothness conditions will also be found.

In Chapter 3, similar results to those obtained in Chapter 2 will be found for wavelets with exponential decay. In particular spline wavelets on $L^2[0,1)$ and on $L^2(\mathbf{R})$ will be used as examples. These wavelets were developed by J.O. Strömberg [St], G. Battle [Ba] and P.G. Lemarié [Le].

Finally, in Chapter 4 the Gibbs effect for wavelet expansions will be explored. The Gibbs effect is the phenomenon in which a partial sum expansion of a function overshoots or undershoots the original function near a jump discontinuity of the function. A condition to determine if there is a Gibbs effect for a general wavelet expansion will be given. This condition will then be used to prove that a Gibbs effect does exist for at least some compactly supported wavelets. Finally, results from a computer analysis will be given which were used to estimate the size of the Gibbs effects for expansions of functions by some compactly supported wavelets. The specific examples used in this chapter deal with the Haar system and I. Daubechies's compactly supported wavelets [Da1].

Results from the work of others will be listed as propositions. New results given in this dissertation will be given as lemmas, theorems and corollaries. Definitions in this paper are commonly known and are the work of others. A good reference for many of the definitions used in this paper is [Fr-Ja-We].

Chapter 1

Introduction to Wavelets

1.1 Motivation, definitions, and examples

Wavelet bases were introduced in the 1980's as new orthonormal bases of various spaces. To motivate the introduction of these new bases, it is useful to look first at the classical Fourier series of a function.

Let

$$f \in L^2(T^1)\,,$$

where T^1 is the circle group and $L^2(T^1)$ is the space of square integrable, one-periodic functions. Then, $f(t)$ can be written in terms of the orthonormal basis $\{e^{i2\pi nt}\}_{n\in\mathbf{Z}}$,

$$f(t) = \sum_{n\in\mathbf{Z}} c_n e^{i2\pi nt}\,, \tag{1.1}$$

where

$$c_n = \int_0^1 f(t)e^{-i2\pi nt}\,dt \tag{1.2}$$

are the Fourier coefficients.

This series is widely used, but it does have its drawbacks. One disadvantage arises in applying the theory to decompose or to reconstruct an actual function, say, for example, an audio signal. In practice one can only use a finite number of the coefficients in (1.1). Cutting off the remaining terms of the series produces an error in the reconstructed signal. The number of Fourier coefficients needed to reconstruct a signal to a desired accuracy may be great, requiring large amounts of computer time and storage. For a Fourier series, a change in the frequency of a signal is not a problem, but a local change in the signal can cause serious problems. The Fourier series cannot represent local changes in a function efficiently.

What is desired is to have a basis which can represent a function to a high degree of accuracy while using a minimal number of terms. Various ways to decompose a function will be presented. This will motivate the idea of the wavelet basis. After that, a wavelet basis will be defined and examples of such bases, which will be used in this dissertation, will be given.

M. Frazier and B. Jawerth [Fr-Ja1], [Fr-Ja2], [Fr-Ja3], and [Fr-Ja-We] found an atomic decomposition theorem for functions in the homogeneous Triebel-Lizorkin spaces, $\dot{F}_p^{\alpha,q}$, and the homogeneous Besov spaces, $\dot{B}_p^{\alpha,q}$. Here the above spaces are defined as $\dot{F}_p^{\alpha,q} = \{f \in \mathcal{S}' : \|f\|_{\dot{F}_p^{\alpha,q}} = \left\| \{\sum_{\nu\in\mathbf{Z}} (2^{\nu\alpha} |\varphi_{2^{-\nu}} * f|)^q\}^{1/q} \right\|_{L^p} < \infty\}$ and $\dot{B}_p^{\alpha,q} = \{f \in \mathcal{S}' : \|f\|_{\dot{B}_p^{\alpha,q}} = \{\sum_{\nu\in\mathbf{Z}} (2^{\nu\alpha} \|\varphi_{2^{-\nu}} * f\|_{L^p})^q\}^{1/q} < \infty\}$ for any function $\varphi \in \mathcal{S}$ (functions with rapid decay, defined in Definition (2.0.1)), such that $\operatorname{supp} \hat{\varphi} \subset [-1/2, -2] \bigcup [1/2, 2]$ and $|\hat{\varphi}(\xi)| \geq c > 0$ if $3/5 \leq |\xi| \leq 5/3$, and for $\alpha \in \mathbf{R}$,$0 < p, q \leq \infty$, $p \neq \infty$ for $\dot{F}_p^{\alpha,q}$. (Note: $\dot{F}_p^{0,2} \sim L^p$ for $1 < p < \infty$, $\dot{F}_p^{0,2} \sim H^p$ for $0 < p \leq 1$, and $\dot{B}_\infty^{\alpha,\infty} \sim \mathrm{Lip}(\alpha)$ for $0 < \alpha < 1$.) For M. Frazier and B. Jawerth's results, they used the following proposition.

Proposition 1.1.1 *Fix $N \in \mathbf{Z}_+$. Then there exist functions θ, and $\varphi \in \mathcal{S}(\mathbf{R}^n)$ such that*

$$\operatorname{supp}\theta \subset \{x : |x| \le 1\}\,, \tag{1.3}$$

$$\int x^\gamma \theta(x)dx = 0 \qquad \textit{if } |\gamma| \le N\,, \tag{1.4}$$

$$\operatorname{supp}\hat{\varphi} \subset \{\xi : \tfrac{1}{2} \le |\xi| \le 2\}\,, \tag{1.5}$$

$$|\hat{\varphi}(\xi)| \ge c > 0 \qquad \textit{if } \tfrac{3}{5} \le |\xi| \le \tfrac{5}{3}\,, \tag{1.6}$$

$$\sum_{\nu\in\mathbf{Z}} \hat{\theta}(2^\nu\xi)\hat{\varphi}(2^\nu\xi) = 1 \qquad \textit{for } \xi \in \mathbf{R}^n - \{0\}\,. \tag{1.7}$$

From this proposition, a version of the Calderón formula,

$$f = \sum_{\nu\in\mathbf{Z}} \varphi_{2^{-\nu}} * \theta_{2^{-\nu}} * f\,, \tag{1.8}$$

is obtained and is used to derive the following atomic decomposition:

Proposition 1.1.2 *Suppose $\alpha \in \mathbf{R}^n$, $0 < p, q < \infty$, and $N \in \mathbf{Z}_+$. If $f \in \dot{B}_p^{\alpha,q}$ $(\dot{F}_p^{\alpha,q})$, then there exists a sequence $s = \{s_Q\}_Q \in \dot{b}_p^{\alpha,q}$ $(\dot{f}_p^{\alpha,q})$ and smooth N– atoms $\{a_Q\}_Q$ such that*

$$f = \sum_Q s_Q a_Q \tag{1.9}$$

and $\|s\|_{\dot{b}_p^{\alpha,q}} \le c\,\|f\|_{\dot{B}_p^{\alpha,q}} \quad (\|s\|_{\dot{f}_p^{\alpha,q}} \le c\,\|f\|_{\dot{F}_p^{\alpha,q}})$.

A similar result exists for the inhomogeneous spaces.

Remark. Here $Q = Q_{\nu,k} = \{x \in \mathbf{R}^n : 2^{-\nu}k_i \le x_i \le 2^{-\nu}(k_i+1), i = 1, 2, ...n\}$, $\dot{f}_p^{\alpha,q} = \{s_Q : \|s\|_{\dot{f}_p^{\alpha,q}} = \left\|(\sum_Q [|Q|^{-\alpha/n-1/2}\,|s_q|\,\chi_Q]^q)^{1/q}\right\|_{L^p} < \infty\}$, and $\dot{b}_p^{\alpha,q} = \{s_Q : \|s\|_{\dot{b}_p^{\alpha,q}} = \{\sum_{\nu\in\mathbf{Z}}(\sum_{\ell(Q)=2^{-\nu}}[|Q|^{-\alpha/n-1/2+1/p}\,|s_Q|]^p)^{q/p}\}^{1/q} < \infty\}$, where $\alpha \in \mathbf{R}$,$0 < p, q \le \infty$, and $p \ne \infty$ for $\dot{f}_p^{\alpha,q}$.

Remark. Decomposition theorems related to the Bergman reproducing formula instead of the Calderón formula can be found in [Co-Ro]. Here the building blocks are molecules (see [Co-We] and [Ta-We]).

REMARK. A function a_Q is called a *smooth N-atom* if it satisfies the following conditions:

$$\{a_Q\} \in \mathcal{D}(\mathbf{R}^n)\,, \tag{1.10}$$

$$\operatorname{supp} a_Q \subset 3Q\,, \tag{1.11}$$

$$\int_{\mathbf{R}^n} x^\nu a_Q(x)\,dx = 0 \qquad \text{if } |\nu| \le N\,, \tag{1.12}$$

and

$$\sup_{x\in\mathbf{R}^n} |D^\gamma a_Q(x)| \le c_\gamma \ell(Q)^{-|\gamma|-\frac{n}{2}} \qquad \text{for } \gamma \in \mathbf{Z}^n_+\ . \tag{1.13}$$

What is to be noted in the decomposition (1.9) is that the atoms depend on the function f. Namely,

$$a_Q(x) = \frac{1}{s_Q}\int_Q \theta_{2^{-\nu}}(x-y)(\varphi_{2^{-\nu}} * f)(y)\,dy \tag{1.14}$$

The next step is to find a decomposition in which the decomposing functions are independent of the function f. This was obtained with the $\varphi-transform$ of M. Frazier and B. Jawerth [Fr-Ja1], [Fr-Ja2] , [Fr-Ja3], and [Fr-Ja-We]. Using some similar ideas to those used above, they started with the following result:

Proposition 1.1.3 *Suppose* $\varphi \in \mathcal{S}(\mathbf{R}^n)$, *satisfies* $\operatorname{supp}\hat\varphi \subset \{\xi : \frac{1}{2} \le |\xi| \le 2\}$ *and* $|\hat\varphi(\xi)| > c > 0$ *if* $\frac{3}{5} \le |\xi| \le \frac{5}{3}$ *(see Definition (2.0.1)). Then there exists a* $\psi \in \mathcal{S}(\mathbf{R}^n)$ *satisfying the same conditions above, such that*

$$\sum_{\nu\in\mathbf{Z}} \overline{\hat\varphi(2^{-\nu}\xi)}\hat\psi(2^{-\nu}\xi) = 1 \qquad \textit{for } \xi \ne 0\,. \tag{1.15}$$

From this, the following Calderón formula is obtained:

$$f = \sum_{\nu\in\mathbf{Z}} \tilde\varphi_{2^{-\nu}} * \psi_{2^{-\nu}} * f\,, \tag{1.16}$$

where

$$\tilde\varphi_{2^{-\nu}}(x) = \overline{\varphi_{2^{-\nu}}(-x)}\,.$$

Applying an extension of the Shannon formula to (1.16), the following decomposition is obtained:

$$f = \sum_Q \langle f, \varphi_Q \rangle \, \psi_Q \, . \tag{1.17}$$

This decomposition holds for the spaces indicated in Proposition (1.1.2). The $\varphi - transform$ of f is defined as $\mathcal{S}_\varphi f = \{\langle f, \varphi_Q \rangle\}_Q$. What is important to note here is the φ_Q's and ψ_Q's are independent of the function f. It has been shown that in order for an expansion like that of (1.17) to be orthonormal, $|\mathrm{supp}\, \hat{\varphi}|$ must be greater than or equal to $(2\pi)^n$, see [Fr-Ja-We]. Since this is not the case for the above decomposition, the decomposition of (1.17) is not orthonormal. This decomposition does however have many properties similar to an orthonormal system. For a more detailed discussion of this, see [Fr-Ja-We].

By applications of the Calderón formulas, decompositions were obtained **(1)** in terms of atoms in $\mathcal{D}(\mathbf{R}^n)$ which depend on the function being evaluated, and **(2)** in terms of the functions in $\mathcal{S}(\mathbf{R}^n)$, which are independent of the function being evaluated and which gives a decomposition similar in many ways to an orthonormal decomposition.

The above work is related to the notion of wavelets developed by P.G. Lemarié and Y. Meyer [Le-Me], where a function $f \in L^p \, , 0 < p < \infty$, has a decomposition in terms of wavelets which live in $\mathcal{S}(\mathbf{R}^n)$, are orthonormal in L^2, and are independent of f.

Before discussing these and other wavelets, a wavelet basis will be defined. A **wavelet basis** is an orthonormal basis of smooth functions generated by dilations by 2^{-m} and translations by $n2^{-m}$ of a single function called a mother function. Here, m and n are both integers.

Example. The Haar system illustrates the general structure of wavelets although it is not a family of continuous functions. Even though it is not, strictly speaking, a wavelet basis, we will consider it at various times because of its similarity to wavelets, and because of its relative simplicity. In this case, the mother function is defined by

$$h(x) = \chi_{[0,\frac{1}{2})}(x) - \chi_{[\frac{1}{2},1)}(x)\,.$$

The elements of the basis are found by the dilations and translations of h as described above. That is,

$$h_{m,n}(x) = 2^{m/2} h(2^m x - n)\,,$$

and $\{h_{m,n}\}_{m,n\in\mathbf{Z}}$ is an orthonormal basis of $L^2(\mathbf{R})$. This basis has nice local properties, but it is undesirable when smoothness plays a role. What is preferred is to have wavelets which have the general structure of the Haar system and also satisfy a regularity condition.

The three following examples of wavelets will be used. **(1)** As mentioned above, P.G. Lemarié and Y. Meyer [Le-Me] constructed wavelets in $\mathcal{S}(\mathbf{R}^n)$. **(2)** J.-O. Strömberg [St] developed spline wavelets while looking for an unconditional basis for the Hardy space H^p. G. Battle [Ba] and P.G. Lemarié [Le] developed these bases in the context of wavelets. The spline wavelets have exponential decay, but only C^N smoothness (to any finite degree N). **(3)** I. Daubechies [Da1] constructed compactly supported wavelets with C^N smoothness. The support of these wavelets increased with the smoothness; to have C^∞ smoothness, wavelets must have infinite support. These examples will be discussed in much greater detail in the following chapters. In the rest of this paper, we will restrict our work to one dimension.

1.2 Underlying structure for wavelet bases

A general structure, called a multiresolution analysis, for wavelet bases in $L^2(\mathbf{R})$ was described by S. Mallat [Ma].

Let

$$\dots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \dots$$

be a family of closed subspaces in $L^2(\mathbf{R})$ where

$$\bigcap_{m\in\mathbf{Z}} V_m = \{0\}\,, \qquad \overline{\bigcup_{m\in\mathbf{Z}} V_m} = L^2(\mathbf{R})$$

and

$$f \in V_m \Longleftrightarrow f(2\cdot) \in V_{m+1}\,.$$

Then there exists a $\phi \in V_0$ such that $\{\phi_{m,n}\}_{n\in\mathbf{Z}}$ is an orthonormal basis of V_m, where

$$\phi_{m,n}(x) = 2^{m/2}\phi(2^m x - n)\,. \tag{1.18}$$

Define W_m such that $V_{m+1} = V_m \oplus W_m$. Thus, $L^2(\mathbf{R}) = \sum \oplus W_m$. Then there exists a $\psi \in W_0$ such that $\{\psi_{m,n}\}_{n\in\mathbf{Z}}$ is an orthonormal basis of W_m, and $\{\psi_{m,n}\}_{m,n\in\mathbf{Z}}$ is a **wavelet basis of** $L^2(\mathbf{R})$,where

$$\psi_{m,n}(x) = 2^{m/2}\psi(2^m x - n)\,. \tag{1.19}$$

The function ϕ is called the **father function**, and ψ is called the **mother function**.

Remark. For $f \in L^2(\mathbf{R})$, the projection map of $L^2(\mathbf{R})$ onto V_m is

$$\Pi_m : L^2(\mathbf{R}) \to V_m$$

defined by

$$\begin{aligned} \Pi_m f(x) &= \sum_{j=-\infty}^{m-1} \sum_{k=-\infty}^{\infty} \langle f, \psi_{j,k}\rangle\, \psi_{j,k}(x) \\ &= \sum_{n\in\mathbf{Z}} \langle f, \phi_{m,n}\rangle\, \phi_{m,n}(x)\,, \end{aligned}$$

and $\Pi_m f(x)$ will be called a **dyadic sum** of f. Also, a **general partial sum** of f will be defined by a projection of $L^2(\mathbf{R})$ *into* V_{m+1}, namely,

$$\mathcal{S}_m^{l\sigma} : L^2(\mathbf{R}) \to V_{m+1}$$

is defined by

$$\mathcal{S}_m^{l\sigma} f(x) = \sum_{j=-\infty}^{m-1} \sum_{k=-\infty}^{\infty} \langle f, \psi_{j,k} \rangle \, \psi_{j,k}(x) + \sum_{k=0}^{l} \left\langle f, \psi_{m,\sigma(l)} \right\rangle \psi_{m,\sigma(l)}(x) \, ,$$

where σ is some permutation of the integers. These two sums will be used throughout this paper.

Example. To illustrate this structure, consider the Haar system. In this case, $V_0 = \{f \in L^2 : f \text{ is constant on } [n, n+1), n \in \mathbf{Z}\}$ and $V_m = \{f \in L^2(\mathbf{R}) : f \text{ is constant on } [2^{-m}n, 2^{-m}(n+1)), n \in \mathbf{Z}\}$. $\phi(x) = \chi_{[0,1)}$, and $\psi(x) = \chi_{[0,\frac{1}{2})}(x) - \chi_{[\frac{1}{2},1)}(x)$.

By adding a regularity condition to the wavelet structure, the following result can be obtained.

Proposition 1.2.1 *Let $\phi(x)$ satisfy the conditions for a multiresolution analysis. Also, let $\phi(x)$ be regular; that is, there exists a constant c such that $|\phi(x)|$, $|\phi'(x)| \leq c/(1+|x|^2)$ for all $x \in \mathbf{R}$. Define $K(x,y) = \sum_{n \in \mathbf{Z}} \phi(x-n)\phi(y-n)$. Then $\int_{\mathbf{R}} K(x,y)\, dy = 1$.*

Proof. Let $\alpha(x) = \int_{\mathbf{R}} K(x,y)\, dy$, a one-periodic function. Since ϕ is regular, $\alpha(x) \in L^\infty(\mathbf{R})$.

Let $f(t) = \chi_{[-1,1]}(x)$, and define $\Pi_m : L^2(\mathbf{R}) \longrightarrow V_m$ to be the projection map. Suppose $x \in [-\frac{1}{2}, \frac{1}{2}]$.

Then,

$$\Pi_m f(x) \quad = \quad \sum_{n \in \mathbf{Z}} \left(\int_{\mathbf{R}} f(y) \phi_{m,n}(y)\, dy \right) \phi_{m,n}(x)$$

$$
\begin{aligned}
&= 2^m \int_{-1}^{1} K(2^m x, 2^m y)\, dy \\
&= \int_{-2^m}^{2^m} K(2^m x, y)\, dy \\
&= \alpha(2^m x) + O(2^{-m}).
\end{aligned}
$$

By the multiresolution analysis conditions, $\Pi_m f(x) \longrightarrow 1$ in $L^2([-\frac{1}{2}, \frac{1}{2}])$ as $m \to \infty$.

Since $\alpha(2^m x)$ is 2^{-m}-periodic, this implies that $\alpha(x) = 1$ almost everywhere.

To show that $\alpha(x) = 1$ everywhere, it now suffices to show that $\alpha(x)$ is continuous.

$$\alpha(x) = \sum_{n \in \mathbf{Z}} \phi(x - n) \int_{\mathbf{R}} \phi(t)\, dt\,.$$

$\int_{\mathbf{R}} \phi(t)\, dt$ is a constant, so it remains to show that $\sum_{n \in \mathbf{Z}} \phi(x - n)$ is continuous. Since $\sum_{n \in \mathbf{Z}} \phi(x - n)$ is 1-periodic, x can be restricted to $(-\frac{1}{2}, \frac{1}{2})$. Continuity can now be shown if it can be demonstrated that if $\lim_{j \to \infty} x_j = x$, then

$$\lim_{j \to \infty} \sum_{n \in \mathbf{Z}} \phi(x_j - n) = \sum_{n \in \mathbf{Z}} \phi(x - n) \tag{1.20}$$

for x_j's restricted to $(-1, 1)$.

Using the Dominated Convergence Theorem, (1.20) can be proved, since ϕ regular implies that ϕ is continuous and that

$$|\phi(x_j - n)| < g(n) = \begin{cases} c & n = 0 \\ 2c/n^2 & n \neq 0 \end{cases} .$$

This completes the proof. □

This result will be used in Chapters 2 and 3.

1.3 Localization properties for wavelets

In the last section, it was mentioned that the Haar system has nice localization properties but lacks smoothness. In this dissertation, pointwise convergence conditions of wavelets which possess desirable smoothness conditions will be given. To motivate the idea of wavelets having good pointwise properties, an example of a local property of wavelets will be examined in this section. Frequency information localized in time for the wavelet transform will be compared to the corresponding information given by the classical Fourier transform and the windowed Fourier transform.

Let $f(t) \in L^1(\mathbf{R})$ be a function in time. The Fourier transform of f is defined as

$$\hat{f}(\xi) = \int_{\mathbf{R}} f(t) e^{-i\xi t}\, dt \tag{1.21}$$

which gives the frequency content of f. The problem is that the frequency information is not given locally with respect to time.

One method to improve the information is to use a windowed Fourier transform

$$\mathcal{C}_{mn}(f) = \int_{\mathbf{R}} f(t) g(t - nt_0) e^{-im\xi_0 t}\, dt\,, \tag{1.22}$$

where g is a "windowing function", which acts as a cut off function, and the transform gives information about f in an area about time nt_0. The problem with this transform is that the windowed area it looks at in time is fixed once the function g is chosen. There is no way to increase or decrease the region in time being examined once the windowed function is fixed.

A better method for this type of analysis is to use a wavelet transform

$$c_{mn}(f) = \int_{\mathbf{R}} f(t) \psi_{m,n}(t)\, dt\,, \tag{1.23}$$

where $\psi_{m,n}$ is a wavelet basis element. Since $\psi_{m,n}(t) = 2^{m/2}\psi(2^m x - n)$ becomes "essentially narrower" with dilations in m, the wavelet transform is able to zoom in on any sized interval in time and give the frequency content for that region in time. Hence, of all of the above transforms, the wavelet transform can best describe the frequency content locally in time. A more detailed discussion of this is given in [Da2].

Chapter 2

Pointwise convergence for wavelets with rapid decay

In this chapter, various pointwise convergence properties will be presented for wavelets with rapid decay. Let us begin by formally stating the following definition which has already been informally used in Chapter 1.

Definition 2.0.1 *A function g has* **rapid decay**, $g \in \mathcal{S}(\mathbf{R})$*, if and only if* $g \in C^{\infty}(\mathbf{R})$ *and* $\sup |x^n D^m g| < \infty$ *for all* n, m. *(Note:* $g \in \mathcal{S}(\mathbf{R})$ *implies that* $|g(x)| \leq C_N/(1+|x|^N)$ *for all* $N \geq 0$ *as* $|x| \to \infty$*).*

The condition $|\phi(x)|, |\psi(x)| \leq C_N/(1+|x|)^N$ for all N, which is a consequence of Definition (2.0.1), will also be used. In some of our results, ϕ and ψ are also required to have some vanishing moments.

One example of wavelets which satisfy these conditions are the wavelets of P.G. Lemarié and Y. Meyer [Le-Me]. A brief discussion of these wavelets will be given in the first section. In the second section, some preliminary work is done that will be used in the final section where convergence results will be proven.

2.1 Wavelets in $\mathcal{S}(\mathbf{R})$

In this section, the wavelets in $\mathcal{S}(\mathbf{R})$ of P.G. Lemarié and Y. Meyer will be discussed briefly. More details about these wavelets can be found in [Fr-Ja-We] and [Le-Me].

The proposition of P.G. Lemarié and Y. Meyer which demonstrates the existence of these wavelets is the following.

Proposition 2.1.1 *There exists a real valued function $\psi \in \mathcal{S}(\mathbf{R})$ such that the collection $\{\psi_{m,n}(x)\}_{m,n\in\mathbf{Z}} = \{2^{m/2}\psi(2^m x - n)\}_{m,n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{R})$. The Fourier transform of ψ has compact support,*

$$\operatorname{supp}\hat{\psi} \subset [-\frac{8}{3}\pi, -\frac{2}{3}\pi] \cup [\frac{2}{3}\pi, \frac{8}{3}\pi],$$

and all moments of ψ vanish,

$$\int_{\mathbf{R}} x^j \psi(x)\, dx = 0 \qquad \textit{for all } j \geq 0.$$

Remark. With this system of wavelets, it can be shown that a father function $\phi \in \mathcal{S}(\mathbf{R})$ can be defined by

$$\hat{\phi}(\xi) = \begin{cases} \cos\omega(\xi), & \text{for } 0 \leq |\xi| \leq \frac{4\pi}{3} \\ 0, & \text{otherwise} \end{cases}$$

where ω is a function which will be discribed below. This definition shows that ϕ has vanishing moments of all orders strictly greater than zero, $\int_{\mathbf{R}} x^\alpha \phi(x)\, dx = 0$ for $\alpha > 0$.

To construct these wavelets, P.G. Lemarié and Y. Meyer started with a nondecreasing, real valued odd function $\theta \in C^\infty$, where $\theta(\xi) = \frac{\pi}{4}$ for $\xi \geq \frac{\pi}{3}$. They let ω be an even function where

$$\omega(\xi) = \begin{cases} 0, & \text{for } 0 \leq \xi \leq \frac{2\pi}{3} \quad \text{and for } \xi \geq \frac{8\pi}{3} \\ \frac{\pi}{4} + \theta(\xi - \pi) & \text{for } \frac{2\pi}{3} \leq \xi \leq \frac{4\pi}{3} \\ \frac{\pi}{4} - \theta(\frac{\xi}{2} - \pi) & \text{for } \frac{4\pi}{3} \leq \xi \leq \frac{8\pi}{3}. \end{cases}$$

Then, ψ is defined by

$$\hat{\psi}(\xi) = e^{-i\xi/2} \sin \omega(\xi).$$

We see that ψ is given by

$$\psi(x) = \frac{1}{\pi} \int_0^\infty \cos[(x - \frac{1}{2})\xi] \sin \omega(\xi)\, d\xi.$$

In [Fr-Ja-We] and [Le-Me], it is proved that this constructed function ψ satisfies the conditions of Proposition (2.1.1).

Another construction of these wavelets can be found in [Au-We-Wi1]. In that paper, a local sine and cosine basis of R.R. Coifman and Y. Meyer is used as the building blocks for these wavelets. The local sine and cosine bases are motivated by the windowed Fourier transform which takes the Fourier transform and cuts it off sharply by a multiplication with a characteristic function. The bases of Coifman and Meyer have local properties similar to the windowed Fourier transform, but they also have arbitrarily smooth cut-offs. With these functions developed, a very natural construction for Lemarié and Meyer's wavelets is obtained.

An important fact about this wavelet basis is the following proposition of P.G. Lemarié and Y. Meyer (see [Le-Me]).

Proposition 2.1.2 *The wavelet basis* $\{\psi_{m,n}\}_{m,n\in\mathbf{Z}}$ *defined in Proposition (2.1.1) is an unconditional basis for* $L^p(\mathbf{R}), 1 < p < \infty$, $BMO(\mathbf{R})$ *and* $H^1(\mathbf{R})$.

Remark. This proposition is proved for $\mathbf{R}^n$ in [Le-Me].

This is all of the information that is needed on these wavelets to move on to the convergence results for this chapter.

2.2 Kernel estimates

Now that a brief discussion has been given on wavelets in $\mathcal{S}(\mathbf{R})$, a size estimate for the kernel of their dyadic sum expansion will be found. This estimate will be used in the next section to obtain various convergence results.

Let $f \in L^p$, $1 \leq p \leq \infty$. Using the notation of Section (1.2), the dyadic sum projection can be written as

$$\Pi_m : L^p \to V_m$$
$$\Pi_m f(x) = \sum_{j=-\infty}^{m-1} \sum_{k=-\infty}^{\infty} \langle f, \psi_{jk} \rangle \, \psi_{jk}(x).$$

In order to write an expression for $\Pi_m f(x)$ with only one summation, the above can be written in terms of an expansion of functions generated by the father function, namely,

$$\Pi_m f(x) = \sum_{n \in \mathbf{Z}} \langle f, \phi_{mn} \rangle \, \phi_{mn}(x) \tag{2.1}$$

Because of the size condition on ϕ, (2.1) can be rewritten in the form

$$\Pi_m f(x) = \int_{\mathbf{R}} f(y) K_m(x, y) \, dy \tag{2.2}$$

where,

$$K_m(x, y) = \sum_{n \in \mathbf{Z}} \phi_{m,n}(x) \phi_{m,n}(y). \tag{2.3}$$

What is of interest is how $\Pi_m f(x)$ behaves as m tends to infinity. Specifically, we wish to compute $\lim_{m \to \infty} |\Pi_m f(x) - f(x)|$. Using Proposition (1.2.1),

$$|\Pi_m f(x) - f(x)| = \left| \int_{\mathbf{R}} [f(y) - f(x)] K_m(x, y) \, dy \right| . \tag{2.4}$$

In order to obtain estimates for the limit of (2.4) as m tends to infinity, under various situations, a size estimate will be computed for $|K_m(x, y)|$.

By definition,

$$\begin{aligned}|K_m(x,y)| &= \left|\sum_{n\in\mathbf{Z}} 2^{m/2}\phi(2^m - n)2^{m/2}\phi(2^m y - n)\right| \\ &\leq 2^m \sum_{n\in\mathbf{Z}} |\phi(2^m x - n)|\,|\phi(2^m y - n)|\,.\end{aligned}$$

Because of the rapid decay condition on ϕ,

$$|K_m(x,y)| \leq 2^m C_N \sum_{n\in\mathbf{Z}} \frac{1}{1+|2^m x - n|^N}\,\frac{1}{1+|2^m y - n|^N} \tag{2.5}$$

for all $N \in \mathbf{Z}^+$.

To estimate the above expression, it is helpful to first study the function g defined by

$$g(t) = \frac{1}{[(1+|2^m x - t|)(1+|2^m y - t|)]^N}. \tag{2.6}$$

For $t \neq 2^m y$, $2^m x$,

$$\frac{dg(t)}{dt} = \frac{-N[(1+|2^m x - t|)\operatorname{sgn}(2^m y - t) + (1+|2^m y - t|)\operatorname{sgn}(2^m x - t)]}{[(1+|2^m x - t|)(1+|2^m y - t|)]^{N+1}}. \tag{2.7}$$

Without loss of generality, let $y \leq x$. From the above expression, $g'(t) = 0$ if and only if $2^m y < t < 2^m x$ and $(1+|2^m x - t|) = -(1+|2^m y - t|)$. Thus,

$$\frac{dg(t)}{dt} = 0 \qquad \Longleftrightarrow \qquad t = 2^m\frac{x+y}{2}$$

Thus, the critical points of $g(t)$ are $2^m y$,$2^m(x+y)/2$,and $2^m x$. Also, from (2.7), it can be determined that $g(t)$ is increasing on $(-\infty, 2^m y) \cup (2^m(x+y)/2, 2^m x)$ and decreasing on $(2^m y, 2^m(x+y)/2) \cup (2^m x, \infty)$. It is also easy to see that $g(t)$ is symmetric about $2^m(x+y)/2$.

With this information gathered about the function g, (2.5) can now be estimated.

$$|K_m(x,y)| \leq 2^m C_N \left\{\int_{\mathbf{R}} g(t)\,dt + 2[g(2^m y) + g(2^m x)]\right\}.$$

Because of the symmetry of g, the above expression reduces to

$$\begin{aligned} |K_m(x,y)| &\leq 2^m C_N \left\{ \int_{-\infty}^{2^m(x+y)/2} \frac{1}{(1+|2^m x - t|)^N (1+|2^m y - t|)^N}\, dt \right. \\ &\quad \left. + \frac{4}{1+2^m |x-y|^N} \right\}. \end{aligned} \tag{2.8}$$

It now remains to estimate the integral in the above expression.

$$\begin{aligned} &\int_{-\infty}^{2^m(x+y)/2} \frac{1}{(1+|2^m x - t|)^N (1+|2^m y - t|)^N}\, dt \\ &\leq \left\{ \sup_{t\in(-\infty, 2^m(x+y)/2]} \frac{1}{(1+|2^m x - t|)^N} \right\} \int_{-\infty}^{2^m(x+y)/2} \frac{1}{(1+|2^m y - t|)^N}\, dt \\ &= \frac{1}{(1+2^m|x-y|/2)^N} \left\{ \int_{-\infty}^{2^m y} \frac{1}{[1+(2^m y - t)]^N}\, dt \right. \\ &\quad \left. + \int_{2^m y}^{2^m(x+y)/2} \frac{1}{[1+(t-2^m y)]^N}\, dt \right\} \\ &= \frac{2^N}{(2+2^m|x-y|)^N} \left\{ \int_0^\infty \frac{1}{(1+u)^N}\, du + \int_0^{2^m(x-y)/2} \frac{1}{(1+u)^N}\, du \right\} \\ &\leq \frac{C_N}{(1+2^m|x-y|)^N} \qquad \leq \frac{C_N}{1+(2^m|x-y|)^N} \qquad \text{for } N>1 \end{aligned}$$

Thus, from (2.8) and the above computation, the following result follows.

Theorem 2.2.1 *Let* $K_m(x,y) = \sum_{n\in\mathbf{Z}} \phi_{m,n}(x)\phi_{m,n}(y)$*, where* ϕ *has rapid decay* ($\phi(x) \leq C_N/(1+|x|^N)$ *for* $N \in \mathbf{Z}^+$*). Then,*

$$|K_m(x,y)| \leq C_N \frac{2^m}{1+(2^m|x-y|)^N} \qquad \leq C_N 2^m \tag{2.9}$$

for $N > 1$.

With this theorem, various convergence results about wavelet expansions can be proved.

2.3 Convergence results

Now that an estimate has been obtained for $|K_m(x,y)|$, this estimate can be used with equation (2.4) to determine convergence properties for dyadic sums of wavelet expressions. The first result that will be proved is the following.

Theorem 2.3.1 *For $f \in L^p(\mathbf{R})$, $1 \le p \le \infty$, x in the Lebesgue set of f, and $|\phi(x)| \le \frac{C_N}{1+|x|^N}$ for some $N > 1$ (in particular, if $\phi \in \mathcal{S}(\mathbf{R})$),*

$$\Pi_m f(x) = \sum_{n \in \mathbf{Z}} \langle f, \phi_{m,n} \rangle \, \phi_{m,n}(x) \quad \longrightarrow \quad f(x) \qquad \text{as } m \to \infty.$$

In particular, $\Pi_m f \to f$ almost everywhere. Furthermore, if f is bounded and uniformly continuous, then $\Pi_m f \to f$ uniformly.

Proof. By the definition of a Lebesgue point, if x is in the Lebesgue set of f then for all $\delta > 0$ there exists a $\eta > 0$ such that

$$\int_{|x-y| \le r} |f(y) - f(x)| \, dy \le \delta r, \qquad \text{for all } r \le \eta.$$

Choose any $\delta > 0$ and let η be as defined above. From (2.4),

$$\begin{aligned} |\Pi_m f(x) - f(x)| &= \left| \int_{\mathbf{R}} [f(y) - f(x)] K_m(x,y) \, dy \right| \\ &\le I_1 + I_2 \,, \end{aligned} \tag{2.10}$$

where $I_1 = \left| \int_{|x-y|<\eta} [f(y) - f(x)] K_m(x,y) \, dy \right|$
and $I_2 = \left| \int_{|x-y|\ge\eta} [f(y) - f(x)] K_m(x,y) \, dy \right|$.
Let us first examine I_1.

$$\begin{aligned} I_1 &= \left| \int_{|x-y|<\eta} [f(y) - f(x)] K_m(x,y) \, dy \right|, \\ &\le A_1 + A_2 \end{aligned} \tag{2.11}$$

where $A_1 = \int_{|x-y| \le 2^{-m}} |f(y) - f(x)| \, |K_m(x,y)| \, dy$,
and $A_2 = \int_{2^{-m} \le |x-y| < \eta} |f(y) - f(x)| \, |K_m(x,y)| \, dy$.

A_1 is easy to estimate. By Theorem (2.2.1), for m sufficiently large

$$\begin{aligned} A_1 &\leq C_N 2^m \int\limits_{|x-y|\leq 2^{-m}<\eta} |f(y)-f(x)|\, dy \\ &\leq C_N \delta\,. \end{aligned} \tag{2.12}$$

In estimating A_2, it can be assumed again that m is large enough so that the interval of integration is nonempty. Also, it can be assumed that $\eta < 1$. Then, $\frac{\eta}{2^m} \leq \frac{1}{2^m} < \eta$ and there exists an l', $0 < l' \leq m-1$, such that $2^{-l'-1}\eta \leq 2^{-m} < 2^{-l'}\eta$. Then,

$$\begin{aligned} A_2 &= \int\limits_{\eta/2^{l'+1}\leq|x-y|<\eta/2^{l'}} |f(y)-f(x)|\,|K_m(x,y)|\, dy \\ &\quad + \sum_{l=0}^{l'-1} \int\limits_{\eta/2^{l+1}\leq|x-y|<\eta/2^{l}} |f(y)-f(x)|\,|K_m(x,y)|\, dy \\ &\leq \sum_{l=0}^{l'} \int\limits_{\eta/2^{l+1}\leq|x-y|<\eta/2^{l}} |f(y)-f(x)|\,|K_m(x,y)|\, dy\,. \end{aligned}$$

Using the size estimate on $|K_m(x,y)|$,

$$\begin{aligned} A_2 &\leq \sum_{l=0}^{l'} \int\limits_{\eta/2^{l+1}\leq|x-y|<\eta/2^{l}} |f(y)-f(x)|\, \frac{C_N 2^m}{1+(2^m\,|x-y|)^N}\, dy \\ &< \sum_{l=0}^{l'} \frac{C_N 2^m}{1+(2^{m-l-1}\eta)^N} \int\limits_{\eta/2^{l+1}\leq|x-y|<\eta/2^{l}} |f(y)-f(x)|\, dy \\ &\leq C_N \sum_{l=0}^{l'} \frac{2^m}{1+(2^{m-l-1}\eta)^N}\delta\eta/2^l\,, \end{aligned}$$

since x is a Lebesgue point of f. Thus,

$$A_2 < C_N\delta \sum_{l=-l'}^{0} \frac{1}{1+(2^{m+l-1})^N} 2^{m+l}\eta\,. \tag{2.13}$$

Now, to deal with the sum in (2.13), let $t_l = 2^{m+l-1}\eta$. Then, $t_{-l'} = 2^{m-l'-1}\eta > 0$ and $t_0 = 2^{m-1}\eta$. Also, let $\Delta_{l'} = t_{-l'} - 0 = 2^{m-l'-1}\eta$, and for $l \neq l'$, $\Delta_l = t_l - t_{l-1} =$

$2^{m+l-2}\eta$. Thus, $\Delta_l \geq C2^{m+l}\eta$,for $-l' \leq l \leq 0$. With this notation and (2.13), we have the following:

$$A_2 < C_N\delta \sum_{l=-l'}^{0} \frac{1}{1+{t_l}^N}\Delta_l \,.$$

Since $1/(1+{t_l}^N)$ decreases as l increases,

$$A_2 < C_N\delta \int_0^\infty \frac{1}{1+t^N}\,dt \leq C_N\delta \qquad \text{for } N > 1\,. \tag{2.14}$$

From (2.11), (2.12) and (2.14), we have that

$$I_1 < C_N\delta\,, \tag{2.15}$$

where δ is arbitrarily small.

Now, we need to estimate I_2. Recall that

$$I_2 = \left| \int\limits_{|x-y|\geq\eta} [f(y)-f(x)]K_m(x,y)\,dy \right| .$$

Let $\chi(y) = \chi_{\{y:|x-y|\geq\eta\}}(y)$. Then,

$$I_2 = \left| \int_{\mathbf{R}} [f(y)-f(x)]\chi(y)K_m(x,y)\,dy \right| .$$

If $0 < p < \infty$, by Hölder's inequality

$$I_2 \leq \|f\|_p \, \|\chi(\cdot)K_m(x,\cdot)\|_{p'} + |f(x)| \, \|\chi(\cdot)K_m(x,\cdot)\|_1\,,$$

where $1/p + 1/p' = 1$. The norms in (2.3) will now be estimated. First,

$$\begin{aligned} \|\chi(\cdot)K(x,y)_m(x,\cdot)\|_1 &= \int\limits_{|x-y|\geq\eta} |K_m(x,y)| \; dy \\ &\leq C_N 2^m \int\limits_{|x-y|\geq\eta} \frac{1}{(1+2^m\,|x-y|)^N}\,dy \end{aligned}$$

$$
\begin{aligned}
&= C_N 2^m \left\{ \int_{-\infty}^{x-\eta} \frac{1}{[1+2^m(x-y)]^N}\, dy \right. \\
&\qquad \left. + \int_{x+\eta}^{\infty} \frac{1}{[1+2^m(y-x)]^N}\, dy \right\} \\
&= C_N 2^m \int_{\eta}^{\infty} \frac{1}{(1+2^m t)^N}\, dt \\
&= \frac{C_N}{(1+2^m\eta)^{N-1}} \qquad \text{for } N > 1 \qquad (2.16)
\end{aligned}
$$

Secondly,

$$
\begin{aligned}
\|\chi(\cdot)K_m(x,\cdot)\|_{p'} &= \left(\int\limits_{|x-y|\ge\eta} |K_m(x,y)|^{p'}\, dy \right)^{1/p'} \\
&\le C_N \left[\int\limits_{|x-y|\ge\eta} \left(\frac{2^m}{(1+2^m|x-y|)^N} \right)^{p'} dy \right]^{1/p'} \\
&= C_N \left[\int_{\eta}^{\infty} \left(\frac{2^m}{(1+2^m t)^N} \right)^{p'} dt \right]^{1/p'} \\
&= C_N \left(\frac{1}{1-Np'} 2^{m(p'-1)} \frac{1}{(1+2^m\eta)^{Np'-1}} \right)^{1/p'} \\
&= C_{Np} \frac{1}{(1+2^m\eta)^{N-1}} \left(\frac{2^m}{1+2^m\eta} \right)^{1/p} \\
&\le C_{Np} \frac{1}{(1+2^m\eta)^{N-1}} \eta^{-1/p}. \qquad (2.17)
\end{aligned}
$$

Combining results from (2.3),(2.16), and (2.17) yields

$$
I_2 \le \|f\|_p C_{Np} \frac{1}{(1+2^m\eta)^{N-1}} \eta^{-1/p} + |f(x)|\, C_N \frac{1}{(1+2^m\eta)^{N-1}}. \qquad (2.18)
$$

If $p = 1$,

$$
I_2 \le C_N \|f\|_1 \frac{1}{(1+2^m\eta)^{N-1}} \eta^{-1}, \qquad (2.19)
$$

and if $p = \infty$,

$$
I_2 \le |f|\, C_N \frac{1}{(1+2^m\eta)^{N-1}} \qquad (2.20)
$$

Thus, for η fixed, (see (2.18),(2.19), and 2.20))

$$I_2 \longrightarrow 0 \qquad \text{as } m \to \infty.$$

Now, (2.10) says that $|\Pi_m f(x) - f(x)| \leq I_1 + I_2$. Since we can make I_1 arbitrarily small by choosing η small enough, and $I_2 \longrightarrow 0$ for η fixed as m tends to infinity, we have that

$$|\Pi_m f(x) - f(x)| \longrightarrow 0 \qquad \text{as } m \to \infty. \tag{2.21}$$

Convergence at all Lebesgue points has now been proved. Since it has been shown in the proof that the convergence of $\Pi_m f(x)$ depends only on δ and the value of $f(x)$, this convergence is uniform when the function f is bounded and uniformly continuous. The proof of the theorem is now complete. □

To extend this result to general partial sums, similar ideas are used. Let

$$\mathcal{S}_m^{l\sigma} : L^p \to V_{m+1}$$

be defined by

$$\mathcal{S}_m^{l\sigma} f(x) = \sum_{j=-\infty}^{m-1} \sum_{k\in\mathbf{Z}} \langle f, \psi_{jk}\rangle\, \psi_{jk}(x) + \sum_{k=0}^{l} \left\langle f, \psi_{m,\sigma(k)} \right\rangle \psi_{m,\sigma(k)}(x), \tag{2.22}$$

where σ is a permutation of the integers. Equation (2.22) can also be written in the form

$$\mathcal{S}_m^{l\sigma} f(x) = \int_{\mathbf{R}} f(y) \left\{ \sum_{n\in\mathbf{Z}} \phi_{m,n}(x)\phi_{m,n}(y) + \sum_{k=0}^{l} \psi_{m,\sigma(k)}(x)\psi_{m,\sigma(k)}(y) \right\} dy \tag{2.23}$$

By Proposition (1.2.1) and the fact that $\int_{\mathbf{R}} \psi(y)\, dy = 0$,

$$\begin{aligned}
&\left|\mathcal{S}_m^{l\sigma} f(x) - f(x)\right| \\
&\quad = \left| \int_{\mathbf{R}} [f(y) - f(x)] \left\{ \sum_{n\in\mathbf{Z}} \phi_{m,n}(x)\phi_{m,n}(y) + \sum_{k=0}^{l} \psi_{m,\sigma(k)}(x)\psi_{m,\sigma(k)}(y) \right\} dy \right| \\
&\quad \leq |\Pi_m f(x) - f(x)| + \left| \int_{\mathbf{R}} [f(y) - f(x)] \sum_{k=0}^{l} \psi_{m,\sigma(k)}(x)\psi_{m,\sigma(k)}(y)\, dy \right|.
\end{aligned} \tag{2.24}$$

Equation (2.24) gives us a way to examine general partial sums for wavelet bases.

In the case that x is a Lebesgue point of f, the term $|\Pi_m f(x) - f(x)|$ has already been estimated. Consider the second term.

$$\left| \int_{\mathbf{R}} [f(y) - f(x)] \sum_{k=0}^{l} \psi_{m,\sigma(k)}(x)\psi_{m,\sigma(k)}(y)\, dy \right| \leq$$
$$\int_{\mathbf{R}} |f(y) - f(x)| \left| \sum_{k=0}^{l} \psi_{m,\sigma(k)}(x)\psi_{m,\sigma(k)}(y) \right| dy$$

Now, concentrating on $\left|\sum_{k=0}^{l} \psi_{m,\sigma(k)}(x)\psi_{m,\sigma(k)}(y)\right|$, we see that

$$\begin{aligned} \left| \sum_{k=0}^{l} \psi_{m,\sigma(k)}(x)\psi_{m,\sigma(k)}(y) \right| &\leq \sum_{k=0}^{l} \left|\psi_{m,\sigma(k)}(x)\right| \left|\psi_{m,\sigma(k)}(y)\right| \\ &\leq \sum_{k \in \mathbf{Z}} |\psi_{mk}(x)|\, |\psi_{mk}(y)|\,, \end{aligned}$$

which leads to the same estimate as that made for $|K_m(x,y)|$ in Section 2.2. Thus we can extend the result of Theorem (2.3.1) to obtain the following corollary:

Corollary 2.3.2 *For $f \in L^p(\mathbf{R})$, $1 \leq p \leq \infty$, x in the Lebesgue set of f, and ϕ and ψ in $\mathcal{S}(\mathbf{R})$ (at least $|\phi(x)|$, $|\psi(x)| \leq C_N/(1+|x|^N)$, for some $N > 1$),*

$$\mathcal{S}_m^{l\sigma} f(x) \longrightarrow f(x) \qquad \textit{as } m \to \infty,$$

where $\mathcal{S}_m^{l\sigma} f(x)$ is defined in (2.22).
In particular, $\mathcal{S}_m^{l\sigma} f \to f$ almost everywhere. Furthermore, if f is uniformly continuous, then $\mathcal{S}_m^{l\sigma} f \to f$ uniformly.

Now that almost everywhere convergence has been proved, we will look at rates of pointwise convergence under certain conditions on the function at a point. These results will follow easily from the size estimate of the kernel.

The first pointwise condition that we will consider is the following.

Definition 2.3.3 *$f \in \mathbf{\Lambda}_\alpha(\mathbf{x})$ if and only if there exists a polynomial $\mathcal{P}$ of degree strictly smaller that α such the*

$$|f(y) - \mathcal{P}(y-x)| = O(|x-y|^\alpha).$$

With this property, the following result can be obtained.

Theorem 2.3.4 *For $f \in \Lambda_\alpha(x)$, $|\phi(x)|, |\psi(x)| \leq C_N/(1+|x|^N)$, ψ has at least vanishing moments of orders 0,1,2,...$[\![\alpha]\!]$, and ϕ has vanishing moments of orders 1,2,....$[\![\alpha]\!]$ (in particular, P.G. Lemarié and Y. Meyer's wavelets in $\mathcal{S}(\mathbf{R})$,*

$$\left|\mathcal{S}_m^{l\sigma} f(x) - f(x)\right| \leq C_{N\alpha} 2^{-m\alpha}.$$

where $N > \alpha + 1$ and $\mathcal{S}_m^{l\sigma}$ is defined in (2.22)

Proof. As with the last theorem, the result will be proved for dyadic sums, and the general sum case will carry through as in Corollary (2.3.2). Using (2.4) and the notation of Definition (2.3.3),

$$\begin{aligned} |\Pi_m f(x) - f(x)| &= \left| \int_{\mathbf{R}} \{f(y) - \mathcal{P}(y-x) + \mathcal{P}(y-x) - f(x)\} K_m(x,y)\, dy \right| \\ &\leq \int_{\mathbf{R}} |f(y) - \mathcal{P}(y-x)|\, |K_m(x,y)|\, dy \end{aligned}$$

The last statement is true from the vanishing moments condition since as a function of y, $\mathcal{P}(y-x) - f(x)$ is a polynomial with a constant term of zero. Now, from Definition (2.3.3) and Theorem (2.2.1),

$$\begin{aligned} |\Pi_m f(x) - f(x)| &\leq C_N \int_{\mathbf{R}} |x-y|^\alpha \frac{2^m}{(1+2^m|x-y|)^N}\, dy \\ &\leq C_N 2^m 2^{-m\alpha} \int_{\mathbf{R}} \frac{(2^m|x-y|)^\alpha}{(1+2^m|x-y|)^N}\, dy \\ &= C_N 2^{-m\alpha} \int_0^\infty \frac{t^\alpha}{(1+t)^N}\, dy \\ &\leq C_N 2^{-m\alpha} \qquad \text{for } N > \alpha + 1 \end{aligned}$$

This proves the result. □

REMARK. This theorem can also be proved using the following proposition of S. Jaffard [Ja]

Proposition 2.3.5 *If* $f \in \Lambda_\alpha(x)$, *then*

$$|c_{jk}| = \left| \int_{\mathbf{R}} f(t)\psi_{jk}(t)\,dt \right| \leq C 2^{-(\alpha+1/2)j}(1 + \left|k - 2^j x\right|)^\alpha .$$

This proposition can be used to obtain the desired estimate in Theorem (2.3.4).

$$\begin{aligned}
\left|\mathcal{S}_m^{l\sigma} f(x) - f(x)\right| &= \left| \sum_{j>m} \sum_{k\in\mathbf{Z}} c_{jk}\psi_{jk}(x) + \sum_{k=1}^{l} c_{m,\sigma(k)}\psi_{m,\sigma(k)}(x) \right| \\
&\leq \sum_{j\geq m} \sum_{k\in\mathbf{Z}} |c_{jk}|\,|\psi_{jk}| \\
&\leq C_N \sum_{j\geq m} \sum_{k\in\mathbf{Z}} 2^{-(\alpha+1/2)j} \left(1 + \left|k - 2^j x\right|\right)^\alpha \frac{2^{j/2}}{(1 + |2^j x - k|)^N} \\
&= C_N \sum_{j\geq m} 2^{-\alpha j} \sum_{k\in\mathbf{Z}} \left(1 + \left|k - 2^j x\right|\right)^{-(N-\alpha)} \\
&\leq C_N \sum_{j\geq m} 2^{\alpha j}, \quad \text{for } N > \alpha + 1 \\
&\leq C_N 2^{-\alpha m}
\end{aligned}$$

This also proves Theorem (2.3.4)

The final results which will be presented in this chapter are rates of convergence for wavelet expansions of a function f at a point x with certain conditions on $\omega_f(x,t)$, for small values of t, $t \leq \delta$. Here,

$$\omega_{\mathbf{f}}(\mathbf{x},\mathbf{t}) \equiv \sup\{|f(y) - f(x)| : |x - y| \leq t\}. \tag{2.25}$$

In this work, f will be in L^p, $1 \leq p \leq \infty$.

As in the other results for this section, proofs will be given for the dyadic sum case, since the proof for the general sums carries through in the same way.

Let m be so large that $2^{-m/2} < \delta$. From equation (2.4),

$$|\Pi_m f(x) - f(x)| \leq A + B \tag{2.26}$$

where,

$$A = \left| \int\limits_{|x-y|\le 2^{-m/2}} [f(y) - f(x)] K_m(x,y)\, dy \right|$$

and

$$B = \left| \int\limits_{|x-y|\ge 2^{-m/2}} [f(y) - f(x)] K_m(x,y)\, dy \right|.$$

From (2.9),

$$\begin{aligned}
A &\le C_N 2^m \int\limits_{|x-y|\le 2^{-m/2}} \frac{|f(y)-f(x)|}{(1+2^m\,|x-y|)^N}\, dy \\
&= C_N 2^m \left\{ \int_0^{2^{-m/2}} \frac{|f(x-t)-f(x)|}{(1+2^m t)^N}\, dt + \int_0^{2^{-m/2}} \frac{|f(x+t)-f(x)|}{(1+2^m t)^N}\, dt \right\} \\
&\le C_N 2^m \int_0^{2^{-m/2}} \frac{\omega_f(x,t)}{(1+2^m t)^N}\, dt, \qquad (2.27)
\end{aligned}$$

and from (2.18), for $0 < p < \infty$,

$$\begin{aligned}
B &\le \|f\|_p\, C \frac{1}{(1+2^{m/2})^{N-1}} 2^{m/2p} + |f(x)|\, C_N \frac{1}{(1+2^{m/2})^{N-1}} \\
&= C \left(2^{-m}\right)^{(N-1-1/p)/2} \\
&\le C \left(2^{-m}\right)^{(N-2)/2}, \qquad \text{for } N > 2\,, \qquad (2.28)
\end{aligned}$$

where C depends on N, f, and x. When $p = 1$, or $p = \infty$, (2.19) and (2.20) give the same result as that of (2.28). With the above estimates, rates of convergence can be estimated for specific conditions on $\omega_f(x,t)$.

Theorem 2.3.6 *For $f \in L^p(\mathbf{R})$, $1 \le p \le \infty$, $\omega_f(x,t) = O(t^\alpha)$ for small t, and $|\phi(x)|\,, |\psi(x)| \le C_N/(1+|x|^N)$ for some $N \ge 2\alpha + 2$ (in particular, the wavelets in $\mathcal{S}(\mathbf{R})$ of P.G. Lemarié and Y. Meyer),*

$$\left| \mathcal{S}_m^{l\sigma} f(x) - f(x) \right| \le C(2^{-m})^\alpha,$$

where C depends on N, f, x, p, and α.

PROOF. This theorem will be proved for dyadic sums; the proof for general sums goes through in the same way.

Using (2.26),

$$|\Pi_m f(x) - f(x)| \leq A + B\,,$$

where from (2.27)

$$\begin{aligned} A &\leq C_N 2^m \int_0^{2^{-m/2}} \frac{t^\alpha}{(1+2^m t)^N}\,dt \\ &\leq C_N 2^m (2^{-m})^\alpha \int_0^{2^{-m/2}} \frac{(2^m t)^\alpha}{(1+2^m t)^N}\,dt \\ &\leq C_N (2^{-m})^\alpha \int_0^{2^m/2} \frac{u^\alpha}{(1+u)^N}\,du \\ &\leq C_N (2^{-m})^\alpha \qquad \text{for } N > \alpha+1, \end{aligned}$$

and from (2.28),

$$B \leq C(2^{-m})^\alpha \qquad \text{for } N \geq \alpha + 2.$$

This completes the proof. □

Theorem 2.3.7 *Under the same hypothesis of Theorem (2.3.6), except that* $\omega_f(x,t) = O(1/\left|\log t\right|^\alpha)$*,* $\alpha \leq 1$*,*

$$\left|\mathcal{S}_m^{l\sigma} f(x) - f(x)\right| \leq C \frac{1}{(\log 2^m)^\alpha},$$

where C *depends on* N*,* f*,* x*,* p *and* α*. Here, Definition (2.0.1) must hold for at least some* $N > 2$*.*

PROOF. Again, it is enough to prove the result for dyadic sums. As in the proof above,

$$|\Pi_m f(x) - f(x)| \leq A + B,$$

where A is estimated with (2.27).

$$\begin{aligned} A &\leq C_N \int_0^{2^{-m/2}} \frac{2^m / |\log t|^\alpha}{(1+2^m t)^N}\, dt \\ &\leq C_N \frac{1}{|\log 2^{-m/2}|^\alpha} \int_0^{2^{m/2}} (1+u)^{-N}\, du \\ &\leq C_N 2^\alpha \frac{1}{(\log 2^m)^\alpha} \qquad \text{for } N > 1 \\ &\leq C_{N\alpha} \frac{1}{(\log 2^m)^\alpha}. \end{aligned}$$

Comparing this term to B (of 2.28) as m tends to infinity shows that A is the larger term, and the result is proved. □

This competes the discussion for wavelets with rapid decay.

Chapter 3

Pointwise convergence for wavelets with exponential decay

In Chapter 2, pointwise convergence properties were found for wavelets with rapid decay. In this chapter, similar results will be found for wavelets with exponential decay. In the previous chapter, P.G. Lemarié and Y. Meyer's wavelets in $\mathcal{S}(\mathbf{R})$ were used as examples. In this chapter, we shall look at spline wavelets both in $L^2[0,1)$ and in $L^2(\mathbf{R})$. Since wavelets on $\mathbf{R}$ were studied in the last chapter, in this chapter we will concentrate predominantly on periodic wavelets in order to illustrate the technique in this situation. With this program in mind, spline wavelets in $L^2[0,1)$ will now be introduced.

3.1 The structure of spline wavelets on $L^2[0,1)$

In Section 1.2, a general structure for wavelets defined on the line was given. Periodic wavelets have a slightly different structure. The multiresolution structure for wavelets in $L^2[0,1)$ is the following. Let

$$V_0 \subset V_1 \subset V_2 \subset ......$$

be a family of closed subspaces dense in $L^2[0,1)$, where

$$f \in V_m \Leftrightarrow f(2\cdot) \in V_{m+1}.$$

There exists a function ϕ defined on $\mathbf{R}$ such that $\{\phi_{m,n}(x)\}_{n=0}^{2^m-1}$ is an orthonormal basis of V_m, where $\phi_{m,n}(x) = 2^{m/2}\sum_{l\in\mathbf{Z}}\phi(2^m[x+l]-n)$ is 1-periodic.
Define W_m by

$$W_0 \equiv V_0, \qquad \text{and } V_{m+1} = V_m \oplus W_{m+1} \text{ for } m \geq 0.$$

Then, there exists a function ψ defined on $\mathbf{R}$ such that $\{1\} \cup \{\psi_{m,n}\}_{m=0}^{\infty}{}_{n=0}^{2^m-1}$ is an orthonormal basis of $L^2[0,1)$, where $\psi_{m,n}(x) = 2^{m/2}\sum_{l\in\mathbf{Z}}\psi(2^m[x+l]-n)$ is 1-periodic.

For the spline wavelets, the V_m spaces have a very clear geometric meaning. Let B be the linear subspace of $L^2[0,1)$ of all continuous one-periodic functions. Then, for spline wavelets of order N,

$$\begin{aligned} V_m^N &= \{f \in B : f \text{ is } C^{N-1} \text{ on } \mathbf{R} \text{ and coincides with a polynomial} \\ &\qquad \text{of degree } N \text{ on } [(l-1)/2^m, l/2^m],\ l = 1, 2, ..., 2^m\} \end{aligned}$$

To construct the father and mother functions for the spline wavelets, the following facts will be useful. These facts were shown to this author by J. Soria [So].

Proposition 3.1.1 *For any $N \in \mathbf{N}$, $\sum_{l\in\mathbf{Z}}(\xi+l)^{-(N+1)} = [\pi/\sin\pi\xi]^{N+1} P_N(\xi)$, where P_N is a trigonometric polynomial, $P_N(k) \neq 0$ for all $k \in \mathbf{Z}$, and $P_N(0) = 1$. Also, if N is odd, then $P_N(\xi) \neq 0$ for all $\xi \in \mathbf{Z}$, P_N is 1-periodic, even and positive. If N is even, then $P_N(\xi+1) = -P_N(\xi)$.*

To show that P_N is a trigonometric polynomial, induction is used. Using calculus of residues, it can be show that

$$\sum_{l\in\mathbf{Z}}(\xi+l)^{-2}=\frac{\pi^2}{\sin^2\pi\xi}$$

and that the result holds with $P_1\equiv 1$.
Now assume that the formula is true for $N-1$,

$$\sum_{l\in\mathbf{Z}}(\xi+l)^{-N}=\left(\frac{\pi}{\sin\pi\xi}\right)^N P_{N-1}(\xi),$$

where $P_{N-1}(\xi)$ is a trigonometric polynomial. Taking the derivative of both sides and simplifying,

$$\sum_{l\in\mathbf{Z}}(\xi+l)^{-(N+1)}=\left(\frac{\pi}{\sin\pi\xi}\right)^{N+1} P_N(\xi),$$

where

$$P_N(\xi)=\frac{P'_{N-1}(\xi)\sin(\pi\xi)-N\pi\cos(\pi\xi)P_{N-1}(\xi)}{-N\pi}$$

is clearly a trigonometric polynomial. Verifications of all other statements in the proposition are straightforward.

These P_N functions will be used to construct spline wavelets. Because of the properties of P_N when N is odd, it will turn out that the odd ordered splines are what we will be interested in for this chapter; for the rest of this chapter, N will be an odd number. To construct the mother function ψ^N and the father function ϕ^N for the spline wavelets of order N, the following results will be needed. A good reference for this material is [We].

Proposition 3.1.2 *For* ϕ^N *and* ψ^N *in* $L^2(\mathbf{R})$,
(a) $\xi^{N+1}\hat{\phi}^N(\xi)$ *is 1-periodic if and only if* $\phi^N\in C^{N-1}$ *and coincides with a polynomial of degree less that or equal to* N *on* $[k-1,k]$, $k\in\mathbf{Z}$.
Also,

(b) $\xi^{N+1}\hat{\psi}^N(\xi)$ *is 2-periodic if and only if* $\psi^N \in C^{N-1}$ *and coincides with a polynomial of degree less than or equal to* N *on* $[k-1/2, k/2]$, $k \in \mathbf{Z}$.

Proposition 3.1.3 *For* N *odd,*

(a) $\sum_{l\in\mathbf{Z}} \left|\hat{\phi}^N(\xi+l)\right|^2 = 1$ *if and only if* $\{\phi^N_{m,n}\}_{n=0}^{2m-1}$ *is an orthonormal set,*

and

(b) $\sum_{l\in\mathbf{Z}} \left|\psi^N_{m,n}(\xi+l)\right|^2 = 1$ *if and only if* $\{\psi^N_{jk}\}_{k=0}^{2^j-1}$ *is an orthonormal set.*

Proposition 3.1.4 *For* N *odd,*

(a) if $\psi_{jk} \in W^N_{j+1}$, *then* $\sum_{l\in\mathbf{Z}} \frac{\psi^N(\xi+l)}{(\xi+l)^{N+1}} = 0$ *for* $\xi = k/2^j$, $k \in \mathbf{Z} - \{0\}$,

and

(b) if $\psi^N_{jk} \in V^N_{j+1}$, $\hat{\psi}^N(0) = 0$ *and* $\sum_{l\in\mathbf{Z}} \frac{\hat{\psi}^N(\xi+l)}{(\xi+l)^{N+1}} = 0$, *then* $\psi^N_{jk} \in W^N_{j+1}$.

With these propositions, one gets the following formulas.

$$\hat{\phi}^N(\xi) = (\frac{\sin \pi\xi}{\pi\xi})^{N+1} / \sqrt{P_{2N+1}(\xi)} \tag{3.1}$$

$$\hat{\psi}^N = e^{-i\pi\xi} \frac{(\frac{\sin \pi\xi/2}{\pi\xi/2})^{2(N+1)} (\sin \pi\xi/2)^{N+1}}{P_{2N+1}(\xi/2)\sqrt{\frac{(\sin \pi\xi/2)^{2(N+1)}}{P_{2N+1}(\xi/2)} + \frac{(\cos \pi\xi/2)^{2(N+1)}}{P_{2N+1}((\xi+1)/2)}}} \tag{3.2}$$

Using these two formulas and complex integration techniques, the following result can be proved.

Proposition 3.1.5 *For* N *odd,* $\phi^N(x)$ *and* $\psi^N(x)$ *are* $O(e^{-a|x|})$ *as* $|x| \to \infty$, *for some* $a > 0$.

Finally, it has been shown that these spline wavelets are a basis for $L^p[0,1)$, $1 \le p < \infty$. A detailed discussion of the properties of these various wavelet bases is given in [We].

Other details for the results in this section can be found in [Me] and [We].

3.2 Kernel estimates

We will need a result similar to Proposition (1.2.1) for periodic spline wavelets. Fortunately, this is easy.

Let

$$\mathcal{K}_m(x,y) = 1 + \sum_{j=0}^{m} \sum_{k=0}^{2^j-1} \psi_{jk}(x)\psi_{jk}(y) = \sum_{n=0}^{2^m-1} \phi_{m,n}(x)\phi_{m,n}(y)\,, \tag{3.3}$$

where ψ_{jk} and $\phi_{m,n}$ are the periodic functions defined in Section 3.1, and $\{\psi_{jk}\}_{j,k\in\mathbf{Z}}$ is the set of periodic spline wavelets of some odd order N (The N superscript will be left out unless it is needed in order to simplify the notation.). Let $\mathbf{1}$ be the function which is identically one. Since $\mathbf{1} \in V_0$ for periodic splines,

$$\Pi_m \mathbf{1}(x) = \int_0^1 \mathcal{K}_m(x,y)\,dy = 1. \tag{3.4}$$

From this result, we have the following formula which parallels formula (2.4). Let

$$\Pi_m : L^p \to V_m.$$

For $f \in L^P[0,1)$, equation (3.4) gives us

$$|\Pi_m f(x) - f(x)| = \left| \int_0^1 [f(y) - f(x)]\mathcal{K}_m(x,y)\,dy \right|. \tag{3.5}$$

Similar to our estimates in Section 2.2, we will now estimate $|\mathcal{K}_m(x,y)|$.

Theorem 3.2.1 *Let* $\mathcal{K}_m(x,y) = 1 + \sum_{j=0}^{m} \sum_{k=0}^{2^j-1} \psi_{jk}(x)\psi_{jk}(y)$ $= \sum_{n=0}^{2^m-1} \phi_{m,n}(x)\phi_{m,n}(y)$ *where* $\{\psi_{jk}\}$ *are the periodic spline wavelets of some odd order.*
($\psi_{jk}(x) = 2^{j/2}\sum_{l\in\mathbf{Z}} \psi(2^j[x+l]-k)$, $\phi_{m,n} = 2^{m/2}\sum_{l\in\mathbf{Z}} \phi(2^m[x+l]-n)$*) Since the order of the splines is odd,* $\phi(x)$ *and* $\psi(x)$ *are* $O(e^{-a|x|})$ *as* $|x| \to \infty$ *for some positive number* a *small enough (see Section 3.1). From this it follows that*

$$|\mathcal{K}_m(x,y)| \le C_a 2^m e^{-a2^m\|x-y\|/2},$$

where $\|x\|$ is defined to be the distance from x to the nearest integer.

Proof. Since we are dealing with one-periodic functions, we can assume that $0 \leq x, y < 1$. Let $g_{mn}(x) = \sum_{|l|\geq 1} \phi(2^m[x+l]-n)$. Then $\phi_{m,n}(x) = 2^{m/2}\{\phi(2^m x - n) + g_{mn}(x)\}$. From the exponential decay of ϕ,

$$\begin{aligned}
|g_{mn}(x)| &= \left|\sum_{l=1}^{\infty} \phi(2^m[x+l]-n) + \sum_{l=-\infty}^{-1} \phi(2^m[x+l]-n)\right| \\
&\leq C\{\sum_{l=1}^{\infty} e^{-a(2^m[x+l]-n)} + \sum_{l=-\infty}^{-1} e^{-a(n-2^m[x+l])}\} \\
&= C\{e^{-a(2^m x-n)} + e^{a(2^m x-n)}\} \sum_{l=1}^{\infty} e^{-a2^m l} \\
&\leq C_a\{e^{-a(2^m x-n)} + e^{a(2^m x-n)}\} e^{-a2^m}.
\end{aligned}$$

Thus,

$$|g_{mn}(x)| \leq C_a\{e^{-a(2^m[x+1]-n)} + e^{a(2^m[x-1]-n)}\}. \tag{3.6}$$

Now, $|\mathcal{K}_m(x,y)|$ can be estimated.

$$\begin{aligned}
\mathcal{K}_m(x,y) &= 2^m \sum_{n=0}^{2^m-1} \{\phi(2^m x-n) + g_{mn}(x)\}\{\phi(2^m y-n) + g_{mn}(y)\} \\
&= 2^m \sum_{n=0}^{2^m-1} \{\phi(2^m x-n) g_{mn}(y) + \phi(2^m y-n) g_{mn}(x) \\
&\quad +\phi(2^m x-n)\phi(2^m y-n) + g_{mn}(x) g_{mn}(y)\}.
\end{aligned}$$

Let

$$I = \sum_{n=0}^{2^m-1} \phi(2^m x-n) g_{mn}(y),$$

$$II = \sum_{n=0}^{2^m-1} \phi(2^m y-n) g_{mn}(x),$$

$$III = \sum_{n=0}^{2^m-1} \phi(2^m x-n)\phi(2^m y-n),$$

and

$$IV = \sum_{n=0}^{2^m-1} g_{mn}(x) g_{mn}(y).$$

Then,

$$|\mathcal{K}_m(x,y)| \leq 2^m(|I| + |II| + |III| + |IV|). \tag{3.7}$$

Now, let us estimate the individual terms.

$$\begin{aligned}
|I| &\leq \sum_{n=0}^{2^m-1} |\phi(2^m x - n)|\,|g_{mn}(y)| \\
&\leq C_a \sum_{n=0}^{2^m-1} \Big\{ |\phi(2^m x - n)|\,[e^{-a(2^m[y+1]-n)} + e^{a(2^m[y-1]-n)}] \Big\} \\
&= C_a \Bigg\{ \sum_{0\leq n<2^m x} e^{-a(2^m x - n)}[e^{-a(2^m[y+1]-n)} + e^{a(2^m[y-1]-n)}] \\
&\quad + \sum_{2^m x \leq n < 2^m} e^{-a(n-2^m x)}[e^{-a(2^m[y+1]-n)} + e^{a(2^m[y-1]-n)}] \Bigg\} \\
&= C_a \Bigg\{ e^{-a2^m(1+x+y)} \sum_{0\leq n<2^m x} e^{2an} + e^{-a2^m(1+x-y)} \sum_{0\leq n<2^m x} 1 \\
&\quad + e^{-a2^m(1-x+y)} \sum_{2^m x\leq n<2^m} 1 + e^{-a2^m(1-x-y)} \sum_{2^m x\leq n<2^m} e^{-2an} \Bigg\} \\
&\leq C_a\{e^{-a2^m(1+x+y)}e^{2a2^m x} + e^{-a2^m(1+x-y)}2^m x \\
&\quad + e^{-a2^m(1-x+y)}2^m(1-x) + e^{-a2^m(1-x-y)}e^{-2a2^m x}\}.
\end{aligned}$$

Simplifying, we get

$$|I| \leq C_a\{e^{-a2^m(1-x+y)} + 2^m x e^{-a2^m(1+x-y)} + 2^m(1-x)e^{-a2^m(1-x+y)} + e^{-a2^m(1+x-y)}\}. \tag{3.8}$$

Similarly,

$$|II| \leq C_a\{e^{-a2^m(1-y+x)} + 2^m y e^{-a2^m(1+y-x)} + 2^m(1-y)e^{-a2^m(1-y+x)} + e^{-a2^m(1+y-x)}\}. \tag{3.9}$$

Moving on,

$$\begin{aligned}
|III| &\leq \sum_{n=0}^{2^m-1} |\phi(2^m x - n)|\,|\phi(2^m y - n)| \\
&\leq C_a \sum_{n=0}^{2^m-1} e^{-a|2^m - n| - a|2^m y - n|}.
\end{aligned}$$

Assuming $x \geq y$,

$$\begin{aligned}
|III| &\leq C_a \left\{ \sum_{0\leq n<2^m y} e^{-a(2^m x-n)-a(2^m y-n)} + \sum_{2^m y\leq n<2^m x} e^{-a(2^m x-n)-a(n-2^m y)} \right. \\
&\quad \left. + \sum_{2^m x\leq n<2^m} e^{-a(n-2^m x)-a(n-2^m y)} \right\} \\
&= C_a \left\{ e^{-a2^m(x+y)} \sum_{0\leq n<2^m y} e^{2an} + e^{-a2^m(x-y)} \sum_{2^m y\leq n<2^m x} 1 \right. \\
&\quad \left. + e^{a2^m(x+y)} \sum_{2^m x\leq n<2^m} e^{-2an} \right\} \\
&\leq C_a \left\{ e^{-a2^m(x+y)} e^{2a2^m y} + e^{-a2^m(x-y)} 2^m(x-y) + e^{a2^m(x+y)} e^{-2a2^m x} \right\} \\
&= C_a \left\{ e^{-a2^m(x-y)} + 2^m(x-y) e^{-a2^m(x-y)} + e^{-a2^m(x-y)} \right\} \\
&= C_a [2 + 2^m(x-y)] e^{-a2^m(x-y)} \qquad \text{for } x \geq y\,.
\end{aligned}$$

Now, for all x,$y \in [0,1)$,

$$|III| \leq C_a [2 + 2^m |x-y|] e^{-a2^m|x-y|}\,. \tag{3.10}$$

Finally,

$$\begin{aligned}
|IV| &\leq \sum_{n=0}^{2^m-1} \left| g_{mn(x)} \right| |g_m n(y)| \\
&\leq C_a \sum_{n=0}^{2^m-1} \left(e^{-a(2^m[x+1]-n)} + e^{-a(n-2^m[x-1])} \right) \left(e^{-a(2^m[y+1]-k)} \right. \\
&\quad \left. + e^{-a(n-2^m[y-1])} \right) \\
&= C_a \left\{ e^{-a2^m(2+x+y)} \sum_{n=0}^{2^m-1} e^{2an} + e^{-a2^m(2+x-y)} \sum_{n=0}^{2^m-1} 1 \right. \\
&\quad \left. + e^{-a2^m(2-x+y)} \sum_{n=0}^{2^m-1} 1 + e^{-a2^m(2-x-y)} \sum_{n=0}^{2^m-1} e^{-2an} \right\} \\
&\leq C_a \left\{ e^{-a2^m(2+x+y)} e^{-2a2^m} + e^{-a2^m(2+x-y)} 2^m \right. \\
&\quad \left. + e^{-a2^m(2-x+y)} 2^m + e^{-a2^m(2-x-y)} \right\}\,.
\end{aligned}$$

Thus,

$$|IV| \leq C_a\left\{e^{-a2^m(x+y)} + e^{-a2^m(2-x-y)} + 2^m e^{-a2^m(2+x-y)} + 2^m e^{-a2^m(2-x+y)}\right\}. \tag{3.11}$$

Combining like terms, so far we have that

$$\begin{aligned}|\mathcal{K}_m(x,y)| \leq\ & C_a 2^m\left\{e^{-a2^m(1-x+y)} + e^{-a2^m(1+x-y)} + 2^m x e^{-a2^m(1+x-y)}\right.\\ & +2^m(x-1)e^{-a2^m(1-x+y)} + 2^m y e^{-a2^m(1+y-x)}\\ & +2^m(1-y)e^{-a2^m(1-y+x)} + e^{-a2^m|x-y|}\\ & +2^m|x-y|e^{-a2^m|x-y|} + e^{-a2^m(x+y)} + e^{-a2^m(2-x-y)}\\ & \left.+2^m e^{-a2^m(2+x-y)} + 2^m e^{-a2^m(2-x+y)}\right\}.\end{aligned} \tag{3.12}$$

To simplify (3.12) further, we can use the following fact which is easily verified. $1-x+y$, $1+x-y$,$|x-y|$, $x+y$, and $2-x-y$ are all greater that or equal to $\|x-y\|$, which is defined in the statement of the theorem to be the distance of $x-y$ to the nearest integer. Using this, (3.12) reduces to

$$\begin{aligned}|\mathcal{K}_m(x,y)| \leq\ & C_a 2^m\left\{e^{-a2^m\|x-y\|} + 2^m x e^{-a2^m(1+x-y)} + 2^m(1-x)e^{-a2^m(1-x+y)}\right.\\ & +2^m y e^{-a2^m(1+y-x)} + 2^m(1-y)e^{-a2^m(1-y+x)}\\ & \left.+2^m|x-y|e^{-a2^m|x-y|} + 2^m e^{-a2^m} e^{-a2^m\|x-y\|}\right\}.\end{aligned} \tag{3.13}$$

Since te^{-t} is a bounded function for $t \geq 0$, (3.13) reduces to

$$\begin{aligned}|\mathcal{K}_m(x,y)| \leq\ & C_a 2^m\left\{e^{-a2^m\|x-y\|} + 2^m x e^{-a2^m(1+x-y)}\right.\\ & +2^m(1-x)e^{-a2^m(1-x+y)} + 2^m y e^{-a2^m(1+y-x)}\\ & \left.+2^m(1-y)e^{-a2^m(1-y+x)} + 2^m|x-y|e^{-a2^m|x-y|}\right\}\\ =\ & C_a 2^m\left\{e^{-a2^m\|x-y\|} + 2^m(1+x-y)e^{-a2^m(1+x-y)}\right.\\ & \left.+2^m(1-x+y)e^{-a2^m(1-x+y)} + 2^m|x-y|e^{-a2^m|x-y|}\right\}.\end{aligned} \tag{3.14}$$

Since x and y belong to $[0,1)$, the exponents in (3.14) are all raised to negative powers, so that (3.14) can be further reduced to

$$|\mathcal{K}_m(x,y)| \leq C_a 2^m \qquad \text{for all } x,y \in [0,1). \tag{3.15}$$

A sharper estimate can be obtained for some values of x and y. To get this estimate, we will return to (3.14). Since te^{-t} is a decreasing function for $t \geq 1$, for $\|x-y\| \geq 1/a2^m$, (3.14) becomes

$$\begin{aligned}
|\mathcal{K}_m(x,y)| &\leq C_a 2^m \{e^{-a2^m\|x-y\|} + 2^m \|x-y\| \, e^{-a2^m\|x-y\|}\} \\
&= C_a 2^m \{e^{-a2^m\|x-y\|} + 2^m \|x-y\| \, e^{-a2^m\|x-y\|/2} e^{-a2^m\|x-y\|/2}\} \\
&\leq C_a 2^m \{e^{-a2^m\|x-y\|} + e^{-a2^m\|x-y\|/2}\} \\
&\leq C_a 2^m e^{-a2^m\|x-y\|/2} \quad \text{for } \|x-y\| \geq 1/a2^m. \qquad (3.16)
\end{aligned}$$

Combining (3.15) and (3.16) gives the result of the theorem. □

Theorem (3.2.1) will be used in the next section to obtain some convergence results for periodic spline wavelets.

3.3 Convergence results

As in Section 2.3, the estimate for $|\mathcal{K}_m(x,y)|$, from Theorem (3.2.1), will now be used with equation (3.5) to obtain some convergence results for periodic spline wavelets.

Theorem 3.3.1 *For $f \in L^1[0,1)$, x in the Lebesgue set of f, and using the periodic spline wavelet expansion of f (notation as in Section 3.1),*

$$\lim_{m\to\infty} \Pi_m f(x) = \sum_{n=0}^{2^m-1} \langle f, \phi_{m,n}\rangle \, \phi_{m,n}(x) = f(x).$$

This shows in particular that $\Pi_m f \to f$ almost everywhere. Furthermore, if f is uniformly continuous, $\Pi_m f \to f$ uniformly.

Proof. This result will be proved for L^p, $0 \le p \le \infty$, instead of L^1, so that the result is easily extended for the noncompact case. This proof is similar to the proof of Theorem (2.3.1). In fact the estimates of Theorem (2.3.1) in the setting of $\mathbf{R}$ implies the corresponding estimate (2.9). The novelty is in the modifications needed for the setting of functions on $[0,1)$. Using the notation from that proof, x being in the Lebesgue set of f implies that for all $\delta > 0$ there exists an $\eta > 0$ such that

$$\int_{|x-y|\le r} |f(y) - f(x)|\, dy \le \delta r \qquad \text{for all } r \le \eta.$$

Following the same format as in (2.10), we have that

$$|\Pi_m f(x) - f(x)| \le I_1 + I_2\,, \tag{3.17}$$

where

$$I_1 = \left| \int_{\substack{y\in[0,1)\\ \|x-y\|<\eta}} (f(y) - f(x))\chi(\cdot)\mathcal{K}_m(x,y)\, dy \right|$$

and

$$I_2 = \left| \int_{\substack{y\in[0,1)\\ \|x-y\|\ge\eta}} (f(y) - f(x))\mathcal{K}_m(x,y)\, dy \right|.$$

Examining I_1,

$$I_1 \le A_1 + A_2, \tag{3.18}$$

where

$$A_1 = \int_{\substack{y\in[0,1)\\ \|x-y\|\le 1/(2^m a)}} |f(y) - f(x)|\, |\mathcal{K}_m(x,y)|\, dy,$$

and

$$A_2 = \int_{\substack{y\in[0,1)\\ 1/(2^m a)\le\|x-y\|<\eta}} |f(y) - f(x)|\, |\mathcal{K}_m(x,y)|\, dy.$$

From Theorem (3.2.1),

$$\begin{aligned} A_1 &\le C_a 2^m \int\limits_{\substack{y\in[0,1)\\ \|x-y\|\le 1/(2^m a)<\eta}} |f(y)-f(x)|\,dy \\ &\le C_a\delta \end{aligned} \tag{3.19}$$

since x is a Lebesgue point of f.

As in the proof of Theorem (2.3.1), for m large enough, there exists an l', $0 < l' \le m-1$, such that $2^{-l'-1}\eta \le 1/(2^m a) < 2^{-l'}\eta$. Then,

$$A_2 \le \sum_{l=0}^{l'} \int\limits_{\substack{y\in[0,1)\\ 2^{-l-1}\eta\le\|x-y\|<2^{-l}\eta}} |f(y)-f(x)|\,|\mathcal{K}_m(x,y)|\,dy.$$

Using the size estimates on $|\mathcal{K}_m(x,y)|$,

$$\begin{aligned} A_2 &\le \sum_{l=0}^{l'} \int\limits_{\substack{y\in[0,1)\\ 2^{-l-1}\eta\le\|x-y\|<2^{-l}\eta}} |f(y)-f(x)|\,C_a 2^m e^{-a2^m\|x-y\|/2}\,dy \\ &\le \sum_{l=0}^{l'} C_a 2^m e^{-a2^m(2^{-l-1}\eta)/2} \int\limits_{\substack{y\in[0,1)\\ 2^{-l-1}\eta\le\|x-y\|<2^{-l}\eta}} |f(y)-f(x)|\,dy \\ &\le \sum_{l=0}^{l'} C_a 2^m e^{-a2^{m-l}\eta/4}\delta 2^{-l}\eta \\ &= C_a\delta \sum_{l=-l'}^{0} e^{-a2^{m+l}\eta/4} a2^{m+l}\eta/4, \end{aligned}$$

since x is a Lebesgue point of f.

Now, define the following: $t_l = a2^{m+l}\eta/4$, $t_{-l'} = a2^{m-l'}\eta/4$ and $t_0 = a2^m\eta/4$. Also, let $\Delta_{-l'} = t_{-l'} - 0$ and $\Delta_l = t_l - t_{l-1}$ for $l > -l'$, we then have that

$$\begin{aligned} A_2 &\le C_a\delta \sum_{l=-l'}^{0} e^{-t_l}\Delta_l \\ &\le C_a\delta \int_0^\infty e^{-t}\,dt \\ &= C_a\delta. \end{aligned} \tag{3.20}$$

Thus, from (3.18), (3.19) and (3.20),

$$I_1 \le C_a\delta. \tag{3.21}$$

To estimate I_2, let $\chi(y) = \chi_{\{y: y\in[0,1), \|x-y\|\ge\eta\}}(y)$. For $0 < p < \infty$, Hölder's inequality gives us

$$I_2 \le \|f\|_p \left\|\chi(\cdot)\mathcal{K}_m(x,\cdot)\right\|_{p'} + |f(x)| \left\|\chi(\cdot)\mathcal{K}_m(x,\cdot)\right\|_1 , \tag{3.22}$$

where $1/p + 1/p' = 1$.

Now,

$$\begin{aligned} \left\|\chi(\cdot)\mathcal{K}_m(x,\cdot)\right\|_1 &= \int_{\substack{y\in[0,1)\\ \|x-y\|\ge\eta}} \left|\mathcal{K}_m(x,y)\right| \, dy \\ &\le C_a 2^m \int_{\substack{y\in[0,1)\\ \eta\le\|x-y\|\le 1/2}} e^{-a2^m\|x-y\|/2} \, dy \, , \end{aligned}$$

by the definition of $\|x-y\|$. Letting $t = 2^m \|x-y\|$,

$$\begin{aligned} \left\|\chi(\cdot)\mathcal{K}_m(x,\cdot)\right\|_1 &\le \int_{2^m\eta}^{2^m/2} e^{-at/2} \, dt \\ &= C_a \left(e^{-a2^m/4} - e^{-a2^m\eta/2}\right) . \end{aligned} \tag{3.23}$$

Now,

$$\begin{aligned} \left\|\chi(\cdot)\mathcal{K}_m(x,\cdot)\right\|_\infty &= \sup_{\substack{y\in[0,1)\\ \|x-y\|\ge\eta}} \left|\mathcal{K}_m(x,y)\right| \\ &\le \sup_{\substack{y\in[0,1)\\ \|x-y\|\ge\eta}} C_a 2^m e^{-a2^m\|x-y\|/2} \\ &= C_a 2^m e^{-a2^m\eta/2}, \end{aligned} \tag{3.24}$$

and,

$$\left\|\chi(\cdot)\mathcal{K}_m(x,\cdot)\right\|_{p'} \le \left\|\chi(\cdot)\mathcal{K}_m(x,y)\right\|_\infty^{1/p} \left\|\chi(\cdot)\mathcal{K}_m(x,y)\right\|_1^{1/p'} \tag{3.25}$$

by Hölder's inequality. The cases for $p = 1$ and $p = \infty$, follow in a similar but simplier way. From (3.22) through (3.25), we have that

$$I_2 \to 0 \qquad \text{as } m \to \infty. \tag{3.26}$$

Finally, since δ is arbitrarily small, (3.17), (3.21) and (3.26) tell us that

$$|\Pi_m f(x) - f(x)| \to 0 \qquad \text{as } m \to \infty.$$

Convergence at Lebesgue points, and hence, convergence almost everywhere has now been proved. Since the convergence of $\Pi_m f(x)$ depends only on δ, this convergence is uniform when the function f is uniformly continuous. The theorem is proved. □

Theorem (3.3.1) can be generalized by looking at partial sums defined as

$$\begin{aligned} \mathcal{S}_m^{l\sigma} f(x) &= \int_0^1 f(y)\,dy + \sum_{j=0}^{m-1} \sum_{k=0}^{2^j-1} \langle f, \psi_{jk} \rangle \psi_{jk}(x) + \sum_{k=0}^{l} \left\langle f, \psi_{m\sigma(k)} \right\rangle \psi_{m,\sigma(k)}(x) \\ &= \Pi_m f(x) + \sum_{k=0}^{n} \left\langle f, \psi_{m,\sigma(k)} \right\rangle \psi_{m,\sigma(k)}(x) \end{aligned} \tag{3.27}$$

where $0 \leq n \leq 2^m - 1$.

Corollary 3.3.2 *For $f \in L^1[0,1)$, x in the Lebesgue set of f, and using the periodic spline wavelet expansion of f (see notation in Section 3.1 and formula (3.27)),*

$$lim_{m\to\infty} \mathcal{S}_m^{l\sigma} f(x) = f(x).$$

This shows that $\mathcal{S}_m^{l\sigma} \to f$ almost everywhere. Moreover, if f is uniformly continuous, $\mathcal{S}_m^{l\sigma} f \to f$ uniformly.

Proof. This proof is similar to the proof of Corollary (2.3.2). Let

$$G_m^{l\sigma}(x,y) = \sum_{k=0}^{l} \psi_{m,\sigma(k)}(x) \psi_{m,\sigma(k)}(y).$$

Then, The same argument used in (3.4) can show that $\int_0^1 (\mathcal{K}_m(x,y) + G_m^{l\sigma}(x,y))\,dy = 1$. Therefore,

$$\begin{aligned}\left|\mathcal{S}_m^{l\sigma} f(x) - f(x)\right| &= \left|\int_0^1 (f(y) - f(x))(\mathcal{K}_m(x,y) + G_m^{l\sigma}(x,y)\,dy\right| \\ &\leq \int_0^1 |f(y) - f(x)|\,|\mathcal{K}_m(x,y)|\;dy \\ &\quad + \int_0^1 |f(y) - f(x)|\left|G_m^{l\sigma}(x,y)\right|\;dy.\end{aligned}$$

The first integral was evaluated as in Theorem (3.3.1). The second integral is done similarly since the same estimates can be found for $\left|G_m^{l\sigma}(x,y)\right|$ as was found for $|\mathcal{K}_m(x,y)|$.

This completes the proof. □

In the three sections above, results for spline wavelets were done for the periodic case. In the final section of this chapter, it will be demonstrated how these results extend to spline wavelets in $L^p(\mathbf{R})$.

3.4 Spline wavelets on $L^p(\mathbf{R})$

In this last section on spline wavelets, we will see how results found for the periodic case carry over easily for spline wavelets on the line. We will also look at the rate of convergence for spline expansions of f at x where $f \in \Lambda_\alpha(x)$; the $\Lambda_\alpha(x)$ spaces were defined in Definition (2.3.3).

To begin with, let us look at the basic set up for spline wavelets on the line. This work parallels the work for periodic spline wavelets discussed in Section 3.1. A discussion of both types of spline wavelets can be found in [Me].

For spline wavelets on the line, we go back to the notation used in Section 1.2. Here, V_m^N is defined as follows:

$$V_m^N = \{f \in L^2(\mathbf{R}) : f \in C^{N-1} \text{and coincides with a polynomial of degree } N \text{ on } [(l-1)/2^m, l/2^m),\ l \in \mathbf{Z}\ \}.$$

Letting ϕ^N and ψ^N be the father and mother functions respectively, Propositions (3.1.2), (3.1.3) and (3.1.4) can be proved in this case, except that in Proposition (3.1.3), n and k run over all integers. From these facts, it can be demonstrated that the same functions ϕ^N and ψ^N generate both the periodic spline wavelets and the spline wavelets on $L^p(\mathbf{R})$.

Now, let $K_m^N(x,y) = \sum_{n\in\mathbf{Z}} \phi_{m,n}(x)\phi_{m,n}(y)$. Since the spline wavelets on the line are generated by the same mother and father functions, the following integrals can be compared.

$$\begin{aligned}
&\int_{\mathbf{R}} K_m^N(x,y)\,dy \\
&\quad = \int_{\mathbf{R}} \sum_{n\in\mathbf{Z}} 2^{m/2}\phi^N(2^m x - n)2^{m/2}\phi^N(2^m y - n)\,dy \\
&\quad = \int_0^1 \sum_{n=0}^{2^m-1} \left(\sum_{l\in\mathbf{Z}} \phi^N(2^m[x+l]-n)\right)\left(\sum_{l'\in\mathbf{Z}} \phi^N(2^m[y+l']-n)\right)\,dy \\
&\quad = \int_0^1 \mathcal{K}_m^N(x,y)\,dy.
\end{aligned}$$

In Section 3.2 the last integral was shown to be equal to 1, so we have that

$$\int_{\mathbf{R}} K_m^N(x,y)\,dy = 1. \tag{3.28}$$

Define $\Pi_m^N f$, in the usual way, as the projection of f into the space V_m^N, then (3.28) gives us that

$$\Pi_m^N f(x) - f(x) = \int_{\mathbf{R}} [f(y)-f(x)]K_m^N(x,y)\,dy. \tag{3.29}$$

To simplify notation, as in the earlier part of this chapter, the N superscript will be left out unless needed for clarity. As in other cases discussed, to get estimates for convergence, we will first obtain an estimate for $|K_m(x,y)|$.

Theorem 3.4.1 *Let $K_m(x,y) = \sum_{n\in\mathbf{Z}} \phi_{m,n}(x)\phi_{m,n}(y)$, where $\{\phi_{m,n}\}_{m,n\in\mathbf{Z}} = \{2^{m/2}\phi(2^m x - n)\}_{m,n\in\mathbf{Z}}$ and ϕ is the father function for the spline wavelets of some odd order N. Then,*

$$|K_m(x,y)| \le C_a 2^m e^{-a2^m|x-y|/2}.$$

Proof. Without loss of generality, let $y \le x$. Then,

$$\begin{aligned}
|K_m(x,y)| &= \left|\sum_{n\in\mathbf{Z}} \phi_{m,n}(x)\phi_{m,n}(y)\right| \\
&= \left|\sum_{-\infty<n<2^m y} \phi(2^m x - n)\phi(2^m y - n)\right. \\
&\quad + \sum_{2^m y\le n<2^m x} \phi(2^m x - n)\phi(2^y - n) \\
&\quad \left. + \sum_{2^m x\le n<\infty} \phi(2^m x - n)\phi(2^m y - n)\right|.
\end{aligned}$$

Because of the exponential properties of ϕ, (see Proposition (3.1.5)),

$$\begin{aligned}
|K_m(x,y)| &\le C2^m \left\{ \sum_{-\infty<n<2^m y} e^{-a(2^m x-n)} e^{-a(2^m y-n)} \right. \\
&\quad + \sum_{2^m y\le n<2^m x} e^{-a(2^m x-n)} e^{-a(n-2^m y)} \\
&\quad \left. + \sum_{2^m x\le n<\infty} e^{-a(n-2^m x)} e^{-a(n-2^m y)} \right\} \\
&= C2^m \left\{ e^{-a2^m(x+y)} \sum_{-\infty<n<2^m y} e^{2an} \right. \\
&\quad \left. + e^{-a2^m(x-y)} \sum_{2^m y\le n<2^m x} 1 + e^{-a2^m(-x-y)} \sum_{2^m x\le n<\infty} e^{-2an} \right\} \\
&\le C_a 2^m \left\{ e^{-a2^m(x+y)} e^{2a2^m y} \right. \\
&\quad \left. + e^{-a2^m(x-y)} 2^m(x-y) + e^{-a2^m(-x-y)} e^{-2a2^m x} \right\}.
\end{aligned}$$

In general, for all x and y,

$$\begin{aligned}|K_m(x,y)| &\leq C_a 2^m \Big\{ e^{-a2^m|x-y|} \\ &\qquad + 2^m |x-y| \, e^{-a2^m|x-y|} + e^{-a2^m|x-y|} \Big\} \\ &\leq C_a 2^m e^{-a2^m|x-y|/2} \, ,\end{aligned}$$

since $2^m |x-y| \, e^{-a2^m|x-y|} = C_a 2^m |x-y| \, e^{-a2^m|x-y|/2} e^{-a2^m|x-y|/2} \leq C_a e^{-a2^m|x-y|/2}$.
□

With this estimate, the following result can be proved.

Corollary 3.4.2 *With the notation of (3.29) and Theorem (3.4.1), for $f \in L^p(\mathbf{R})$, $1 \leq p \leq \infty$, and x in the Lebesgue set of f,*

$$\lim_{m\to\infty} \Pi_m f(x) = f(x).$$

In particular, we have convergence almost everywhere. Furthermore, if f is bounded and uniformly continuous, then $\Pi_m f \to f$ uniformly.

Remark. This result is a direct consequence of Theorem (2.3.1), but the proof is much simpler in this cased.

Proof. This proof is similar to the proof of Theorem (3.3.1). Again, since x is a Lebesgue point of f, for all $\delta > 0$ there exists an $\eta > 0$ such that

$$\int\limits_{|x-y|\leq r} |f(y) - f(x)| \, dy \leq \delta r \qquad \text{for all } r \leq \eta,$$

and

$$|\Pi_m f(x) - f(x)| \leq I_1 + I_2,$$

where

$$I_1 = \left| \int\limits_{|x-y|<\eta} [f(y) - f(x)] K_m(x,y) \, dy \right|$$

and

$$I_2 = \left| \int\limits_{|x-y|\geq\eta} [f(y) - f(x)] K_m(x,y)\, dy \right|.$$

I_1 can be estimated as in the proof of Theorem (3.3.1) with $|x-y|$ replacing $\|x-y\|$. With this one obtains

$$I_1 \leq C_a \delta. \tag{3.30}$$

I_2 still needs to be estimated. This is also similar to the proof in the periodic case. With slight changes in the notation, by (3.22) and (3.25), we need only estimate two norms in order to get an estimate for I_2. The first norm estimate is

$$\begin{aligned} \|\chi(\cdot)K(x,y)_m(x,\cdot)\|_1 &= \int\limits_{|x-y|\geq\eta} |K_m(x,y)|\, dy \\ &\leq C_a \int\limits_{|x-y|\geq\eta} 2^m e^{-a2^m|x-y|/2}\, dy \\ &= C_a \int_{a2^m\eta/2}^{\infty} e^{-t}\, dt \\ &= C_a e^{-2^m \eta}. \end{aligned}$$

The second norm estimate is

$$\begin{aligned} \|\chi(\cdot)K_m(x,\cdot)\|_\infty &= \sup_{|x-y|\geq\eta} |K_m(x,y)| \\ &\leq \sup_{|x-y|\geq\eta} C_a 2^m e^{-a2^m|x-y|/2} \\ &= C_a 2^m e^{-a2^m\eta/2}. \end{aligned}$$

Since both of these norms go to zero as m increases, we have that

$$I_2 \to 0 \qquad \text{as } m \to \infty.$$

From this, we have convergence at Lebesgue points, and we have uniform convergence if f is bounded and uniformly continuous. $\square$

The method used to show that the integral for a dyadic sum kernel, $K_m(x,y)$, is one, equation (3.28), can also be used on the integral of a general partial sum kernel to show that it is also one. Then, using the usual arguments for extending results on dyadic sums to general partial sums, we have the following result.

Corollary 3.4.3 *Let*

$$\mathcal{S}_m^{l\sigma} f(x) = \sum_{j=-\infty}^{m-1} \sum_{k\in\mathbf{Z}} \langle f, \psi_{jk}\rangle \, \psi_{jk}(x) + \sum_{k=0}^{l} \left\langle f, \psi_{m,\sigma(k)}\right\rangle \psi_{m,\sigma(k)}(x)$$

be the partial sum spline wavelet expansion for $f \in L^p(\mathbf{R})$, $1 \le p \le \infty$, and x in the Lebesgue set of f. Then,

$$\lim_{m\to\infty} \mathcal{S}_m^{l\sigma} f(x) = f(x).$$

In particular, we have almost everywhere convergence. Furthermore, if f is bounded and uniformly continuous, the convergence is uniform.

The final result which will be discussed in this chapter is one involving a convergence rate of a wavelet expansion of a function f at a point x, where $f \in \Lambda_\alpha(x)$. For a definition of Λ_α, see Definition (2.3.3). As in the proof of (2.3.4), ϕ and ψ will stand for ϕ^N and ψ^N respectively, where $N \ge [\![\alpha]\!]$. Using the notation of Definition (2.3.3),

$$\begin{aligned} |\Pi_m f(x) - f(x)| &= \left| \int_{\mathbf{R}} \{f(y) - \mathcal{P}(y-x) + \mathcal{P}(y-x) - f(x)\} K_m(x,y)\, dy \right| \\ &\le \int_{\mathbf{R}} |f(y) - \mathcal{P}(y-x)|\, |K_m(x,y)|\, dy. \end{aligned}$$

The term $\mathcal{P}(y-x) - f(x)$ drops out since as a function of y it is a polynomial of degree less than N whose value at x is zero, and it is being projected into the space V_m^N of piecewise polynomial functions of degree N. We now have the following:

$$|\Pi_m f(x) - f(x)| \le \int_{\mathbf{R}} |y-x|^\alpha \, C_a 2^m e^{-a2^m|x-y|/2}\, dy$$

$$
\begin{aligned}
&\leq \; C_a 2^m \left\{ \int_{-\infty}^{x} (x-y)^{\alpha} e^{-a2^m(x-y)/2}\, dy \right. \\
&\qquad \left. + \int_{x}^{\infty} (y-x)^{\alpha} e^{-a2^m(y-x)/2}\, dy \right\} \\
&= \; 2C_a 2^m \int_0^{\infty} t^{\alpha} e^{-a2^m t/2}\, dt \\
&= \; 2C_a 2^m \Gamma(\alpha+1)/(2^{m-1}a)^{\alpha+1}
\end{aligned}
$$

The above can be expanded slightly, as done through out this paper, to give the following result about general partial sums.

Theorem 3.4.4 *For $f \in \Lambda_\alpha(x)$ and for spline wavelets of degree greater that or equal to $[\![\alpha]\!]$,*

$$
\left| \mathcal{S}_m^{l\sigma} f(x) - f(x) \right| \leq C_{a\alpha} 2^{-m\alpha}
$$

This completes this chapter.

Chapter 4

Gibbs phenomenon

4.1 Introduction

In Chapters 2 and 3, we examined convergence rates for wavelet expansions of a function at points with certain smoothness conditions. In this chapter, we will look at wavelet expansions of a function at points with jump discontinuities.

When a Fourier series is used to approximate a function with a jump discontinuity, an overshoot at the discontinuity occurs. This phenomenon was noticed by A. Michelson [Mi] and explained by J.W. Gibbs [Gi] in 1899.

Let $g(x)$ be a periodic, piecewise smooth function with a jump discontinuity at x_0. For any fixed x_1, not equal to x_0, the partial sums of $g(x)$ at x_1 approach $g(x_1)$. That is, if s_n is the partial sum of g, then

$$\lim_{n\to\infty} s_n(x_1) = g(x_1)\,.$$

However, if x is allowed to approach the discontinuity as the partial sums are taken to the limit, an overshoot, or undershoot, may occur. That is,

$$\lim_{\substack{n\to\infty \\ x_n\to x_0^+}} s_n(x_n) \neq g(x_0^+)$$

and

$$\lim_{\substack{n\to\infty \\ x_n \to x_0^-}} s_n(x_n) \neq g(x_0^-)$$

are possible. This overshoot, or undershoot, is called the **Gibbs phenomenon**.

To illustrate this effect (see [Zy]), one can look at the 2π-periodic function $g(x) = \frac{1}{\pi}(\pi - x)$, for $0 < x < 2\pi$. The partial sums of the Fourier series of $g(x)$ are

$$s_n(x) = \frac{2}{\pi}\sum_{k=1}^{n} \frac{\sin kx}{k} \qquad \text{for } x \neq 2lk, l \in \mathbf{Z}. \tag{4.1}$$

It can be shown that for any $\epsilon > 0$, there exists a $\delta > 0$ such that for $0 \leq x \leq \delta$,

$$s_n(x) \;=\; \frac{2}{\pi}\sum_{k=1}^{n} \frac{\sin kx}{k} \;=\; \frac{2}{\pi}\int_0^{(n+\frac{1}{2})x} \frac{\sin t}{t}\, dt + E_n\,, \tag{4.2}$$

where $|E_n| < \epsilon$. For n large, $\pi/(n+1/2) < \delta$, and (4.2) becomes

$$s_n\left(\pi/(n+1/2)\right) \;=\; \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{t}\, dt + E_n \;=\; 1.1789... \tag{4.3}$$

This overshoot of the partial sums at 0^+ is about 9% relative to the size of the jump discontinuity (which is 2). The same size undershoot will occur at 0^-.

Writing a general piecewise smooth 2π-periodic function in terms of $g(x)$, the following result can be proved about Fourier series.

Proposition 4.1.1 *Let f be a function of bounded variation, 2π-periodic function. At each jump discontinuity x_0 of f, the Fourier series for f will overshoot (undershoot) $f(x_0^+)$ and undershoot (overshoot) $f(x_0^-)$ if $f(x_0^+) - f(x_0^-)$ is positive (negative). The overshoot and undershoot will be approximately 9% of the magnitude of the jump $\left|f(x_0^+) - f(x_0^-)\right|$.*

J. Foster and F.B. Richards [Fo-Ri] ,[Ri] looked at the Gibbs effect for best L^2 first order spline approximations for a function. The second author examined higher order splines. This work was not done in the spline wavelet context, but the results apply to these wavelets. They looked at approximating a function

$$g(x) = \begin{cases} -1, & -1 \leq x < 0 \\ 1, & 0 \leq x < 1 \end{cases}$$

in $L^2[-1,1]$. F.B. Richard numerically calculated the overshoot at at $g(0^+)$ for splines of degree one through seven, and found that the overshoot was larger than that of the Fourier series. He conjectured that this overshoot approaches the Fourier overshoot as the order of the splines goes to infinity. His paper contains some partial results that support his conjecture.

In this chapter, certain conditions on the wavelet kernel will be examined to determine if a Gibbs effect occurs and what magnitude it is. Specific examples are demonstrated with I. Daubechies's compactly supported wavelets, and computer estimates on the size of the overshoot and undershoot were computed for some of the wavelets with small support.

4.2 Compactly supported wavelets

In this section, the construction of compactly supported wavelets will be described briefly, and some of their properties will be presented. These wavelets were constructed by I. Daubechies in [Da1].

Based on a decomposition and reconstruction algorithm of S. Mallat which utilizes the multiresolution structure for wavelets, I. Daubechies extracted the necessary conditions for constructing wavelets from a sequence of numbers $\{h(n)\}$ without reference to a multiresolution analysis. Her construction for wavelets is the following one. The notation in the statement of this proposition varies slightly from hers due to a difference in the definition of the Fourier transform.

Proposition 4.2.1 *Define* $m_0(\xi) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbf{Z}} h(n)e^{-in\xi}$, *where the* $h(n)$*'s satisfy*

$$\sum_{n\in\mathbf{Z}} |h(n)|\,|n|^{\epsilon} < \infty \qquad \text{for some } \epsilon > 0, \tag{4.4}$$

$$\sum_{n\in\mathbf{Z}} h(n) = \sqrt{2}, \tag{4.5}$$

and

$$\sum_{n\in\mathbf{Z}} h(n)h(n+2k) = \delta_{0,k} = \begin{cases} 0, & k \neq 0 \\ 1, & k = 0. \end{cases} \tag{4.6}$$

Also, $m_0(\xi)$ *can be written in the following form:*

$m_0(\xi) = [\frac{1}{2}(1+e^{-i\xi})]^N F(\xi), N \in \mathbf{Z}_+$, *with* $F(\xi) = \sum_{n\in\mathbf{Z}} f(n)e^{-in\xi}$, *where*

$$\sum_{n\in\mathbf{Z}} |f(n)|\,|n|^{\epsilon} < \infty \qquad \text{for some } \epsilon > 0, \tag{4.7}$$

and

$$\sup_{\xi\in\mathbf{R}} |F(\xi)| < 2^{N-1}. \tag{4.8}$$

Then define $g(n) = (-1)^n h(-n+1)$, *and let* $\hat{\phi}(\xi) = \prod_{j=1}^{\infty} m_0(2^{-j}\xi)$,

$$\phi(x) = \lim_{l\to\infty} \eta_l(x), \tag{4.9}$$

where

$$\eta_l(x) = \sqrt{2}\sum_{n\in\mathbf{Z}} h(n)\eta_{l-1}(2x-n) \qquad \text{and} \qquad \eta_0(x) = \chi_{[-\frac{1}{2},\frac{1}{2})}(x). \tag{4.10}$$

Also, define $\psi(x) = \sqrt{2}\sum_{n\in\mathbf{Z}}(-1)^n h(-n+1)\phi(2x-n)$. *Then, the set of* $\phi_{m,n}(x) = 2^{m/2}\phi(2^m x - n)$ *defines a multiresolution analysis, and the* $\{\psi_{m,n}(x)\}_{m,n\in\mathbf{Z}}$ *is the associated wavelet basis.*

Remark. Condition (4.4) guaranties that $\hat{\phi}$ is well defined; (4.5) makes $\int_{\mathbf{R}} \phi(x)\,dx = 1$; (4.6) gives the orthonormality to the $\{\phi_{m,n}\}$; and (4.7) and (4.8) ensure that ϕ is continuous.

EXAMPLE. If $N = 1$ and $F(\xi) = 1$ in the above proposition, then $\phi(x) = \chi_{[0,1)}(x)$, and one gets the Haar system. In this case, the continuity condition (4.8) fails.

To generate compactly supported wavelets, the following result is used.

Proposition 4.2.2 *$\phi(x)$ and $\psi(x)$ have compact support if and only if only a finite number of the $h(n)$'s in Proposition (4.2.1) are nonzero.*

Thus, to generate compactly supported wavelets, one only needs to find a finite non-zero sequence $\{h(n)\}$ which satisfies the conditions of Proposition (4.2.1). A constructive way to find $F(\xi)$, and hence the $h(n)$'s is presented in [Da1]. In that paper, I. Daubechies also includes values for $\{h(n)\}$ for some of the wavelets with small support. These values are used in computer work for this paper. I. Daubechies also shows that an increase in the number of nonzero $h(n)$'s increases the support size of the wavelets, and increases their smoothness; compactly supported wavelets cannot be C^{∞}. I. Daubechies also showed that unlike the Haar system, no continuous compactly supported wavelets can be symmetric about any point. This fact will be important in the sections that follow.

The following definition of I. Daubechies will be used in the following sections.

Definition 4.2.3 *Let ${}_N\phi$ and ${}_N\psi$ be the function defined by $\{h(n)\}$ satisfying the conditions of Theorem (4.2.1), $h(n) = 0$ for $n < 0$ and $n > 2N - 1$, $h(0) \neq 0$ and $h(2N - 1) \neq 0$.*

I. Daubechies's wavelets come from N values greater than 1. $N = 1$ yields the Haar system. With these assumptions, the support size of ${}_N\phi$ can be determined.

Proposition 4.2.4 *The smallest interval which contains the support of ${}_N\phi$ is $[0, 2N - 1]$.*

REMARK. In this paper, the fact that this is the smallest such interval is needed. The proof of this was not provided in [Da1], so a proof of this will be provided here.

PROOF. From (4.9), it follows that the supp ${}_N\phi \subset [0, 2N - 1]$. It is left to show that this interval is the smallest such interval.
Claim: There does not exist an $\epsilon > 0$ such that ${}_N\phi|_{[0,\epsilon]} \equiv 0$.
Assume there does exist such an ϵ, and let ϵ_0 be the largest such ϵ; a largest must exist since ${}_N\phi$ is continuous. By (4.9),

$$ {}_N\phi(x) = \sqrt{2} \sum_{n=0}^{2N-1} h(n)\, {}_N\phi(2x - n)\,, \tag{4.11} $$

and for $x \leq \min(\epsilon_0, 1/2 + \epsilon_0/2)$,

$$ 0 = {}_N\phi(x) = \sqrt{2} h(0) \phi(2x)\,. \tag{4.12} $$

Since $h(0) \neq 0$, (4.12) implies that $\phi(x) \equiv 0$ for $x \leq \min(2\epsilon_0, 1 + \epsilon_0)$. This violates the maximality of ϵ_0, hence there does not exist such an ϵ and the claim is true.
To show that there does not exist an $\epsilon > 0$ such that ${}_N\phi\,|_{[2N-1-\epsilon, 2N-1]} \equiv 0$, the same procedure can be carried out, using $x \geq 2N - 1 - \min(\epsilon_0, 1/2 + \epsilon_0/2)$ in (4.11) and using the fact that $h(2N - 1) \neq 0$. □

In the following sections, compactly supported wavelets generated by 4, 6, 8, and 10 coefficients will be studied.

4.3 Gibbs phenomenon at the origin

In this work, we will assume that functions are of bounded variation. To begin looking for Gibbs effects, we will first study functions which have a jump discontinuity at zero. A more general setting will be given in Section 4.4. To study the

Gibbs effect of functions with a jump discontinuity at zero, it suffices to look at wavelet expansions of the function

$$f(x) = \begin{cases} -1-x, & -1 \le x < 0 \\ 1-x, & 0 < x \le 1 \\ 0, & \text{else} \end{cases} \tag{4.13}$$

since other functions with a jump discontinuity at zero can be written in terms of f plus a function which is continuous at the origin.

4.3.1 A general formula for dyadic sums

For simplicity, we first restrict attention to the study of dyadic sum wavelet expansions of $f(x)$. Having defined $f(x)$, by (4.13), it is now possible to look at the dyadic sum projections of $f(x)$ into the space V_m, generated by the functions $\{\phi_{m,n}\}$.

$$\Pi_m : L^p(\mathbf{R}) \longrightarrow V_m$$

$$\Pi_m f(x) = \int_{\mathbf{R}} f(y) K_m(x,y)\, dy \tag{4.14}$$

where $K_m(x,y) = 2^m \sum_{n\in\mathbf{Z}} \phi_{m,n}(x)\phi_{m,n}(y)$, and ϕ is the father function. Now, using the definition of f,

$$\begin{aligned}
\Pi_m f(x) &= \int_{-1}^{0} (-1-y) K_m(x,y)\, dy + \int_0^1 (1-y) K_m(x,y)\, dy \\
&= \int_0^1 (-1+t) K_m(x,-t)\, dt + \int_0^1 (1-t) K_m(x,t)\, dt \\
&= \int_0^1 (1-t)\{K_m(x,t) - K_m(x,-t)\}\, dt \\
&= 2^m \int_0^1 (1-t)\{K_0(2^m x, 2^m t) - K_0(2^m x, -2^m t)\}\, dt \\
&= \int_0^{2^m} (1-2^{-m}u)\{K_0(2^m x, u) - K_0(2^m x, -u)\}\, du \\
&= \int_0^{\infty} \chi_{[0,2^m]}(u)(1-2^{-m}u)\{K_0(2^m x, u) - K_0(2^m x, -u)\}\, du\,.
\end{aligned}$$

Since what is of interest is the region about the origin as m tends to infinity, x will be set to $2^{-m}a$, where a is a fixed real number (see remark below). The above expression then becomes

$$\Pi_m f(2^{-m}a) = \int_0^\infty \chi_{[0,2^m]}(u)(1-2^{-m}u)\{K_0(a,u) - K_0(a,-u)\}\,du. \tag{4.15}$$

The absolute value of the argument of the integral is bounded by $|K_0(a,u)| + |K_0(a,-u)|$, which is an integrable function because of the rate of decay of wavelets. Also, the limit of this argument as m tends to infinity is $\chi_{[0,\infty)}(u)\{K_0(a,u) - K_0(a,-u)\}$. Thus, applying the Dominated Convergence Theorem to (4.15), one has

$$\begin{aligned}\lim_{m\to\infty} \Pi_m f(2^{-m}a) &= \int_0^\infty \{K_0(a,u) - K_0(a,-u)\}\,du \\ &= 2\int_0^\infty K_0(a,u)\,du - 1.\end{aligned}$$

Remark. If instead of choosing a as a fixed number, we had a sequence $2^{-m}a_m$, there would be two possibilities: If $a_m \to 0$, then we would end up with equation (4.16) with $a = 0$, and we would have the same expression as if we had chosen $a = 0$. If $a_m \to \infty$, since $2^m a_m$ must tend to zero, a_m must tend to infinity slower that 2^m. Thus, because of the decay conditions of ϕ, the expression of equation (4.15) would tend to zero, and there would be no overshoot. This explains our choice of $x = 2^{-m}a$.

The following theorem has now been obtained.

Theorem 4.3.1 *For f defined in (4.13), $a \in \mathbf{R}$ and using the notation of (4.14)*

$$\lim_{m\to\infty} \Pi_m f(2^{-m}a) = 2\int_0^\infty K_0(a,u)\,du - 1. \tag{4.16}$$

Thus, studying a Gibbs phenomenon reduces to looking at the above integral of the wavelet kernel. Specifically, a Gibbs effect occurs near the origin if and only if

$$\int_0^\infty K_0(a,u)\,du > 1, \qquad \text{for some } a > 0$$

and (or)

$$\int_0^\infty K_0(a,u)\,du < 0, \qquad \textit{for some } a < 0.$$

REMARK. In the following sections, this theorem will be used to study the question of Gibbs effects for the Haar system and for compactly supported wavelets.

4.3.2 Examples

In the last section, results pertain to all wavelets. In this section, we will look at wavelets which have compact support. To see that the dyadic sum of the Haar system expansion for $f(x)$ has no Gibbs effect, one can look at the following integral, where $\phi(x) = \chi_{[0,1)}(x)$ for the Haar system.

$$\begin{aligned}\int_0^\infty K_0(a,u)\,du &= \sum_{n\in\mathbf{Z}} \chi_{[0,1)}(a-n)\int_0^\infty \chi_{[0,1)}(u-n)\,du \\ &= \int_{-[\![a]\!]}^\infty \chi_{[0,1)}(t)\,dt \\ &= \begin{cases} 1, & a \geq 0 \\ 0, & a < 0. \end{cases}\end{aligned}$$

Thus, by Theorem (4.3.1) there is no Gibbs phenomenon in this case.

The existence of a Gibbs effect near the origin for wavelet expansions of f can be proved for certain compactly supported wavelets. To do this, the following lemma is used.

Lemma 4.3.2 *A Gibbs phenomenon for a dyadic wavelet expansion of $f(x)$ generated by the function ${}_N\phi$, defined in Definition (4.2.3), occurs at the right hand side of $f(x)$ if and only if there exists an $a > 0$ such that*

$$\begin{aligned} {}_N\phi(a+1)\int_0^1 {}_N\phi(t)\,dt &+ {}_N\phi(a+2)\int_0^2 {}_N\phi(t)\,dt + \ldots \\ &+ {}_N\phi(a+(2N-2))\int_0^{2N-2} {}_N\phi(t)\,dt < 0. \quad (4.17)\end{aligned}$$

PROOF. From Theorem (4.3.1), there exists a Gibbs effect at the right hand side of the origin if and only if there exists an $a > 0$ such that

$$\int_0^{\infty} K_0(a,u)\,du > 1 \qquad \left(= \int_{\mathbf{R}} K_0(a,u)\,du\right). \tag{4.18}$$

Since the support of ${}_N\phi$ is contained in $[0, 2N-1]$, (4.18) reduces to finding an $a > 0$ such that

$$\sum_{n=-\infty}^{2N-2} {}_N\phi(a+n) \int_n^{2N-1} {}_N\phi(t)\,dt > \sum_{n=-\infty}^{2N-2} {}_N\phi(a+n) \int_0^{2N-1} {}_N\phi(t)\,dt. \tag{4.19}$$

Subtracting the appropriate terms of (4.19) yields (4.17). Thus, the lemma is proved. □

REMARK. For the Haar system, $N = 1$, the left hand side of (4.17) is 0, which implies that there is no Gibbs effect.

We are now able to show that a Gibbs effect does exist for the expansion generated by ${}_2\phi$. From the lemma, showing that a Gibbs phenomenon exists for the wavelet expansion involving ${}_2\phi$ reduces to finding an $a > 0$ such that

$${}_2\phi(a+1) \int_0^1 {}_2\phi(t)\,dt + {}_2\phi(a+2) \int_0^2 {}_2\phi(t)\,dt < 0.$$

To simplify the above expression, let a be greater than or equal to 1. Now an $a \geq 1$ is needed such that

$${}_2\phi(a+1) \int_0^1 {}_2\phi(t)\,dt < 0. \tag{4.20}$$

(4.20) can be verified by showing (1) that $\int_0^1 {}_2\phi(t)\,dt \neq 0$ and (2) that there exists a_1 and $a_2 \geq 1$ such that ${}_2\phi(a_1+1)$ and ${}_2\phi(a_2+1)$ have opposite signs.

Claim 1: $\int_0^1 {}_2\phi(t)\,dt \neq 0$.

PROOF. Assume that $\int_0^1 {}_2\phi(t)\,dt = 0$. Then, integrating (4.11) over $[0,1]$ yields

$$0 = \int_0^1 {}_2\phi(t)\,dt$$

$$
\begin{aligned}
&= \sqrt{2}\{h(0)\int_0^1 {}_2\phi(2t)\,dt + h(1)\int_0^1 {}_2\phi(2t-1)\,dt \\
&\quad + h(2)\int_0^1 {}_2\phi(2t-2)\,dt + h(3)\int_0^1 {}_2\phi(2t-3)\,dt\}
\end{aligned}
$$

Since the support of ${}_2\phi(t)$ is contained in $[0,3]$, the above reduces to

$$
\begin{aligned}
0 &= \sqrt{2}\{h(0)\int_0^1 {}_2\phi(2t)\,dt + h(1)\int_0^1 {}_2\phi(2t-1)\,dt\} \\
&= \frac{\sqrt{2}}{2}\{h(0)\int_0^2 {}_2\phi(t)\,dt + h(1)\int_0^1 {}_2\phi(t)\,dt\} \\
&= \sqrt{2}h(0)\int_0^2 {}_2\phi(t)\,dt
\end{aligned}
$$

by assumption. Since $h(0) \neq 0$ (see [Da1] and Appendix), this implies that $\int_0^2 {}_2\phi(t)\,dt = 0$. Now, integrating (4.11) over $[0,2]$ and using similar arguments shows that $\int_0^3 {}_2\phi(t)\,dt = 0$. This last statement is false since integrating over the whole support of the father function gives the number 1. Thus, the assumption is false and the claim is true. □

Claim 2: There exists numbers $a_1, a_2 \geq 1$ such that ${}_2\phi(a_1+1)$ and ${}_2\phi(a_2+1)$ have opposite signs.

Proof. By Theorem (4.2.4), one can choose an $x \geq 2.5$ such that ${}_2\phi(x) \neq 0$. For such an x, (4.11) becomes

$$
{}_2\phi(x) = \sqrt{2}h(3)_2\phi(2x-3).
$$

Let $a_1 = x-1$ and $a_2 = (2x-3)-1$. Then, since $h(3) < 0$ (see [Da1]), a_1 and a_2 satisfy the claim. □

With these claims, it has been shown that (4.20) is true for some $a \geq 1$, and thus the following result has been proved.

Theorem 4.3.3 *There exists a Gibbs phenomenon on the right hand side of the origin for the dyadic sum wavelet expansion of $f(x)$ generated by ${}_2\phi$.*

This can be made more general. Before doing this, the following lemmas will be needed.

Lemma 4.3.4 *For* ${}_N\phi$*, if* $h(2N-2)+h(2N-1)=\sqrt{2}$*, then* $h(2N-1)=1/\sqrt{2}$*.*

Proof. The hypothesis of the lemma implies that $h^2(2N-2)+2h(2N-2)h(2N-1)$ $+h^2(2N-1)=2$. Equation (4.6) implies that the sum of the first and last term is less than or equal to 1. Thus,

$$h(2N-2)h(2N-1) \geq \frac{1}{2},$$

which by the hypothesis of the lemma can be rewritten as

$$\{h(2N-1)-\frac{1}{\sqrt{2}}\}^2 \leq 0.$$

This statement is only true if $h(2N-1)=1/\sqrt{2}$, so the lemma is proved. □

Lemma 4.3.5 *For* ${}_N\phi$*, if* $h(2N-1)<0$*, then there exists a positive integer* $n<2N-1$ *such that* $\int_0^n {}_N\phi(t)\,dt \neq 0$*.*

Proof. Assume that the lemma is false; that is, assume that $\int_0^n {}_N\phi(t)\,dt=0$ for all integers $n<2N-1$. Then, since $\int_0^{2N-1} {}_N\phi(t)\,dt=1$,

$$\int_{2N-2}^{2N-1} {}_N\phi(t)\,dt = 1 \tag{4.21}$$

and

$$\int_{k-1}^{k} {}_N\phi(t)\,dt = 0 \qquad \text{for } k=0,...,2N-2. \tag{4.22}$$

Using equations (4.21) and (4.22), and integrating equation (4.11) over $[2N-2,2N-1]$, one obtains

$$1 \;=\; \int_{2N-2}^{2N-1} {}_N\phi(t)\,dt$$

$$
\begin{aligned}
&= \sqrt{2}\{h(2N-2)\int_{2N-2}^{2N-1} {}_N\phi(2t-(2N-2))\,dt \\
&\qquad +h(2N-1)\int_{2N-2}^{2N-1} {}_N\phi(2t-(2N-1))\,dt\} \\
&= \frac{\sqrt{2}}{2}\{h(2N-2)\int_{2N-2}^{2N-1} {}_N\phi(t)\,dt \\
&\qquad +h(2N-1)\int_{2N-3}^{2N-1} {}_N\phi(t)\,dt\} \\
&= \frac{\sqrt{2}}{2}\{h(2N-2)+h(2N-1)\}.
\end{aligned}
$$

Thus, $h(2N-2)+h(2N-1)=\sqrt{2}$, and by Lemma (4.3.4) this implies that $h(2N-1)=1/\sqrt{2}$. Since, by the hypothesis, $h(2N-1)<0$, the assumption made in the proof is incorrect, and the lemma is true. □

With these lemmas proved, a more general statement of Proposition (4.3.3) can now be proved.

Theorem 4.3.6 *If $h(2N-1)<0$, then there exists a Gibbs phenomenon on the right hand side of the origin for the dyadic sum wavelet expansion of $f(x)$ generated by ${}_N\phi$.*

Proof. Letting n be the smallest integer to satisfy Lemma (4.3.5), equation (4.17) reduces to looking for an $a>0$ such that

$$
\begin{aligned}
&{}_N\phi(a+n)\int_0^n {}_N\phi(t)\,dt + {}_N\phi(a+(n+1))\int_0^{n+1} {}_N\phi(t)\,dt + \ldots + \\
&\qquad {}_N\phi(a+(2N-2))\int_0^{2N-2} {}_N\phi(t)\,dt < 0. \qquad (4.23)
\end{aligned}
$$

To simplify the above expression, a can be chosen such that $2N-2 \le a+n < 2N-1$. Then, equation (4.23) reduces to

$$
{}_N\phi(a+n)\int_0^n {}_N\phi(t)\,dt < 0. \qquad (4.24)
$$

By the assumption on n, $\int_0^n {}_N\phi(t)\,dt \neq 0$, and (4.24) can be verified if two numbers $x_1, x_2 \in [2N-2, 2N-1]$ can be found such that ${}_N\phi(x_1)$ and ${}_N\phi(x_2)$ have opposite signs.

Choose $x_1 \geq 2N - 1.5$ such that ${}_N\phi(x_1) \neq 0$. Then,

$$
{}_N\phi(x_1) = \sqrt{2}h(2N-1)_N\phi(2x_1 - (2N-1)).
$$

Since $h(2N-1) < 0$, letting $x_2 = 2x_1 - (2N-1)$, the needed numbers x_1 and x_2 have been found. The theorem is now proved. □

Remark. This theorem proves that the Gibbs phenomenon exists for about half of the compactly supported wavelets that I. Daubechies has given the $h(n)$ coefficients for in ([Da1]). The size and the existence of the Gibbs effect for other compactly supported wavelets will be examined below.

So far in this section, Theorem (4.3.1) has been used to prove the existence of a Gibbs phenomenon. In the remaining pages of this section, this theorem will be used to approximate the size of Gibbs effects in some of I. Daubechies compactly supported wavelets. Sizes of Gibbs effects will be approximated by values obtained in FORTRAN programs based on Theorem (4.3.1).

Before beginning to use the computer to assist in this problem, it is necessary to determine where a possible Gibbs phenomenon could occur. To do this, Theorem (4.3.1) will be used to determine where a Gibbs effect could not occur for compactly supported wavelets.

Let

$$
K_0(a,u) = \sum_{n\in\mathbf{Z}} {}_N\phi(a+n)_N\phi(u+n).
$$

For $a > 0$, when is

$$
\int_0^\infty K_0(a,u)\,du \;\; = 1 \qquad \left(= \int_{\mathbf{R}} K_0(a,u)\,du\right) \tag{4.25}
$$

true?

$$\int_0^\infty K_0(a,u)\,du = \sum_{n=-\infty}^{2N-2} {}_N\phi(a+n)\int_n^{2N-1} {}_N\phi(t)\,dt, \tag{4.26}$$

since the support of ${}_N\phi$ is contained in $[0, 2N-1]$. Also,

$$\int_{\mathbf{R}} K_0(a,u)\,du = \sum_{n=-\infty}^{2N-2} {}_N\phi(a+n)\int_0^{2N-1} {}_N\phi(t)\,dt.$$

The two above sums are equal if ${}_N\phi(a+n) = 0$ for $n \geq 1$. This will at least be true if $a+n \geq 2N-1$, and thus, (4.25) is satisfied when $a \geq 2N-2$. Thus, there is no Gibbs effect, as defined in Theorem (4.3.1), for $a \geq 2N-2$ for the wavelet expansion generated by ${}_N\phi$.

Similarly, for $a < 0$, a Gibbs effect will not occur if

$$\int_0^\infty K_0(a,u)\,du = 0. \tag{4.27}$$

As seen from the sum of this integral, equation (4.26), equation (4.27) is true if ${}_N\phi(a+n) = 0$ for $1 \leq n \leq 2N-2$. This is true if $a+n \leq 0$, which implies that there is no Gibbs effect for $a \leq -(2N-2)$.

It has been shown above that in searching for a Gibbs effect of dyadic wavelet expansions of f generated by ${}_N\phi$, one only needs to look in the region
$\{2^{-m}a : a \in (-(2N-2), 2N-2)\}$ as $m \to \infty$.

The next step is to use computer analysis to approximate the value of the integral $\int_0^\infty K_0(a,u)\,du$ for values of a in $[-(2N-2), 2N-2]$. From (4.9), ${}_N\phi(x)$ will be approximated by ${}_N\eta_l(x) = \sqrt{2}\sum_{n=0}^{2N-1} h(n)\,{}_N\eta_{l-1}(2x-n)$ for various values of l, where ${}_N\eta_0(x) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(x)$.

Results from this computer analysis are approximate, but they do give a good idea of the size of the Gibbs effect. For any expansion by compactly supported wavelets, the Gibbs phenomenon on each side of the origin may differ because of the lack of symmetry of these wavelets. This will be reflected in the results given in Table (4.1).

$_N\phi$	$_N\phi \approx {}_N\eta_l$	left side of origin		right side of origin	
		a	$\lim_{m\to\infty} \Pi_m f(2^{-m}a)$	a	$\lim_{m\to\infty} \Pi_m f(2^{-m}a)$
$_2\phi$	$_2\eta_{11}$	-1.1	-1.04	1.0	1.61
$_3\phi$	$_3\eta_8$	-1.0	-1.25	1.0	1.25
$_4\phi$	$_4\eta_7$	-0.9	-1.33	1.6	1.12
$_5\phi$	$_5\eta_7$	-0.8	-1.20	0.8	1.22

Table 4.1: Approximate maximum overshoot and undershoot for dyadic sum wavelet expansions $\lim_{m\to\infty} \Pi_m f(2^{-m}a)$ generated by $_N\phi$

REMARK. For more details on the computer programs and the analysis of the data, see the Appendix.

4.3.3 A general formula for partial sums

In the previous sections, dyadic sums of wavelet terms were examined. In this section and the following one, general wavelet expansions of $f(x)$ will be examined.

Again, $f(x)$ will be defined as in (4.13), and $\mathcal{S}_m^{l\sigma} f(x)$ is defined as its general partial sum, as used in Chapters 2 and 3.

$$\mathcal{S}_m^{l\sigma} f(x) = \Pi_m f(x) + \int_{\mathbf{R}} f(y) G_m^{l\sigma}(x,y)\, dy$$

where

$$G_m^{l\sigma}(x,y) = \sum_{k=0}^{l} \psi_{m,\sigma(k)}(x)\psi_{m,\sigma(k)}(y). \tag{4.28}$$

It follows as in the dyadic sums case that

$$\begin{aligned}\lim_{m\to\infty} \mathcal{S}_m^{l\sigma} f(2^{-m}a) &= \left\{2\int_0^\infty K_0(a,u)\, du - 1\right\} \\ &\quad + \left\{\int_0^\infty G_0^{l\sigma}(a,u)\, du - \int_0^\infty G_0^{l\sigma}(a,-u)\, du\right\}.\end{aligned}$$

Since $\int_{\mathbf{R}} \psi = 0$, the following result is obtained.

Corollary 4.3.7 *For f defined in (4.13), $a \in \mathbf{R}$, and using the notation of (4.28),*

$$\begin{aligned}\lim_{m\to\infty} \mathcal{S}_m^{l\sigma} f(2^{-m}a) &= 2\int_0^\infty K_0(a,u)\,du - 1 + 2\int_0^\infty G_0^{l\sigma}(a,u)\,du \\ &= \lim_{m\to\infty} \Pi_m f(2^{-m}a) + \mathcal{G}^{l\sigma}(a)\end{aligned}$$

The term $\mathcal{G}^{l\sigma}(a)$ gives the value of the limit dependent on which additional terms are added to the dyadic sum. The $\mathcal{G}^{l\sigma}(a)$ term could shift the peak of the Gibbs effect, and could also change the size of it. It is of interest to find which terms are best to include in order to minimize the Gibbs effect. These matters will be treated in some detail in the next section.

4.3.4 Examples

Based on Corollary (4.3.7), the Gibbs phenomenon for the wavelet expansion of f near the origin will be computed for general partial sums.

For the Haar system, it has already been shown that there is no Gibbs effect at the origin for the dyadic sum expansion of f. The next step is to look at the term $\mathcal{G}^{l\sigma}(a)$ of Corollary (4.3.7) for the Haar system.

$$\begin{aligned}\mathcal{G}^{l\sigma}(a) &= 2\int_0^\infty G_0^{l\sigma}(a,u)\,du \\ &= 2\sum_{k=0}^{l} \psi(a+\sigma(k)) \int_0^\infty \psi(u+\sigma(k))\,du \\ &= 2\sum_{k=0}^{l} \psi(a+\sigma(k)) \int_{\sigma(k)}^1 \psi(t)\,dt.\end{aligned}$$

Since $\sigma(k)$ is an integer and $\psi(t) = \chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)}(t)$, the above integral is always zero. Hence, there is no Gibbs phenomenon for partial sum Haar expansions of f.

The next example that will be examined is that of the compactly supported wavelets. Since $\lim_{m\to\infty} \Pi_m f(2^{-m}a)$ has been examined in Section 4.3.2, in this section the term $\mathcal{G}^{l\sigma}(a)$ will be studied. Computer programs have been written to

l	$\sigma(k)$	left side of origin		right side of origin	
		a	$\lim_{m\to\infty} \mathcal{S}_m^{l\sigma} f(2^{-m}a)$	a	$\lim_{m\to\infty} \mathcal{S}_m^{l\sigma} f(2^{-m}a)$
0		-1.01	-1.04	0.99	1.61
1	$\sigma(1) = 0$	-1.01	-1.04	0.99	1.30
1	$\sigma(1) = -1$	-1.01	-1.02	0.99	1.61
2	$\sigma(1) = 0$ $\sigma(2) = -1$	-1.01	-1.02	0.99	1.30

Table 4.2: Approximate maximum overshoot and undershoot for the general partial wavelet sums $\lim_{m\to\infty} \mathcal{S}_m^{l\sigma} f(2^{-m}a)$ generated by ${}_2\phi$ for various values of l and $\sigma(k)$ in $\mathcal{G}^{l\sigma}(a)$ (${}_2\phi$ is approximated by ${}_2\eta_{11}$.)

approximate the value of $\mathcal{G}^{l\sigma}(a)$ for various values of l, σ and a. Before going to the computer, it is now necessary to determine the range of a values that should be examined.

$$\mathcal{G}^{l\sigma}(a) = 2\sum_{k=0}^{l} {}_N\psi(a + \sigma(k)) \int_{\sigma(k)}^{\infty} {}_N\psi(t)\, dt.$$

Since the support of ${}_N\psi$ is $[-\frac{N}{2}, N]$, as shown in ([Da1]), and $\int_{\mathbf{R}} {}_N\psi(t)\, dt = 0$, $\int_{\sigma(k)}^{\infty} {}_N\psi(t)\, dt \neq 0$ implies that $-\frac{N}{2} < \sigma(k) < N$, and $\mathcal{G}^{l\sigma}(a) \neq 0$ implies that $-\frac{N}{2} < a + \sigma(k) < N$. Thus, in looking for values of $\mathcal{G}^{l\sigma}(a)$, the only time that a nonzero value is obtained is when $-N - \frac{N}{2} < a < N + \frac{N}{2}$, and $-\frac{N}{2} < \sigma(k) < N$.

The specific example that has been examined on the computer is the expansion generated by ${}_2\phi$. In this case, $\sigma(k) = 0$ and 1 are the only values of concern. From the above work, when $\sigma(k) = 0$, the values of $-1 < a < 2$ are of interest, and when $\sigma(k) = 1$, the interval $-2 < a < 1$ is what needs to be examined. Thus, the computer programs for this part will compute values for $\lim_{m\to\infty} \mathcal{S}_m^{l\sigma} f(2^{-m}a)$ over the region $-2 < a < 2$; this includes estimating the values for $\mathcal{G}^{l\sigma}(a)$, where $\sigma(k) = 0$ or 1 or both. Results are given in Table (4.2).

In this case, the data suggests that adding additional terms in the partial sum has lessened the Gibbs effect, but that the effect occurs in the same location. In Section 4.4.4, Tables (4.5) and (4.6) will present data that seems to suggest that the location of the maximum overshoot and undershoot may also vary. Further discussion of such data will be given at that time.

4.4 Gibbs phenomenon at a general point

In Section 4.3, the Gibbs phenomenon was examined for the function f which had a discontinuity at the origin. Because of the translation and dilation procedure used to generate wavelets, the expansion of a function is dependent on the actual locations of behaviors of the function; we say that the wavelets are not translation invariant. Because of this fact, it is important to study Gibbs effects for wavelet expansions of functions with a discontinuity at a general point. Therefore, in this section work parallel to that of Section 4.3 will be done for the function

$$g(x) = f(x-b) = \begin{cases} (b+1)-x, & b < x \leq (1+b) \\ (b-1)-x, & b-1 \leq x < b \\ 0, & \text{else} \end{cases} \tag{4.29}$$

which has a discontinuity at the point b.

4.4.1 A general formula for dyadic sums

A dyadic sum expansion for the function g, defined in (4.29) will now be found.

$$\Pi_m : L^p(\mathbf{R}) \to V_m \tag{4.30}$$

$$\Pi_m g(x) = \int_{\mathbf{R}} g(y) K_m(x,y)\, dy$$

$$\begin{aligned}
&= \int_{b-1}^{b} [(b-1)-y] K_m(x,y)\, dy + \int_{b}^{b+1} [(b+1)-y] K_m(x,y)\, dy \\
&= 2^m \{ \int_{b-1}^{b} [(b-1)-y] K_0(2^m x, 2^m y)\, dy \\
&\quad + \int_{b}^{b+1} [(b+1)-y] K_0(2^m x, 2^m y)\, dy \\
&= \int_{2^m(b-1)}^{2^m b} [(b-1) - 2^{-m} t] K_0(2^m x, t)\, dt \\
&\quad + \int_{2^m b}^{2^m(b+1)} [(b+1) - 2^{-m} t] K_0(2^m x, t)\, dt\,.
\end{aligned}$$

As m tends to infinity, points close to b are of interest, so we will let $x = 2^{-m}a + b$, where a is a fixed real number.

$$\begin{aligned}
\Pi_m g(2^{-m}a + b) &= \int_{2^m(b-1)}^{2^m b} [(b-1) - 2^{-m} t] K_0(a + 2^m b, t)\, dt \\
&\quad + \int_{2^m b}^{2^m(b+1)} [(b+1) - 2^{-m} t] K_0(a + 2^m b, t)\, dt.
\end{aligned}$$

Letting $u = -t + 2^m b$ in the first integral, $u = t - 2^m b$ in the second integral, and combining, one gets

$$\Pi_m g(2^{-m}a+b) = \int_0^{2^m} (1-2^{-m}u)\{K_0(a+2^m b, u+2^m b) - K_0(a+2^m b, -u+2^m b)\}\, du. \tag{4.31}$$

Since as m approaches infinity, the $2^m b$ term in the argument of the kernels causes some difficulty, we want to remove the m dependence in the kernels.

The problem of m dependence in the kernels is addressed by examining the definition of the kernel.

$$\begin{aligned}
K_0(x,y) &= \sum_{n \in \mathbf{Z}} \phi(x+n)\phi(y+n) \\
&= \sum_{n \in \mathbf{Z}} \phi(x+n'+n)\phi(y+n'+n),
\end{aligned}$$

where n' is any integer. Thus,

$$K_0(a + 2^m b, u + 2^m b) = K_0(a + 2^m b - [\![2^m b]\!], u + 2^m b - [\![2^m b]\!])$$

and

$$K_0(a+2^m b, -u+2^m b) = K_0(a+2^m b - [\![2^m b]\!], -u+2^m b - [\![2^m b]\!]),$$

where $[\![x]\!]$ is the greatest integer less that or equal to x. Since, as m varies, the value of $2^m b - [\![2^m b]\!]$ may vary, we will restrict m values to a set J such that this expression will be fixed for all m in J. If b is a rational number, there will be a finite number of such sets of m values, if b is irrational, there will be an infinite number of such sets and each will contain one value of m. The notation used is the following:

$$\mathbf{b_J} = 2^m b - [\![2^m b]\!], \qquad \text{for } m \in J. \tag{4.32}$$

Using the convention of (4.32), (4.31) becomes

$$\Pi_m g(2^{-m}a+b) = \int_0^{2^m} (1-2^{-m}u)\{K_0(a+b_J, u+b_J) - K_0(a+b_J, -u+b_J)\}\, du$$

for $m \in J$. Since the m dependence has been taken out of the above kernel's argument, the limit as m tends to infinity can now be taken, as was done in Section 4.3.1.

$$\begin{aligned}\lim_{\substack{m\in J\\ m\to\infty}} \Pi_m g(2^{-m}a+b) &= \int_0^{\infty} \{K_0(a+b_J, u+b_J) - K_0(a+b_J, -u+b_J)\}\, du \\ &= \int_{b_J}^{\infty} K_0(a+b_J, u)\, du - \int_{-\infty}^{b_J} K_0(a+b_J, u)\, du\end{aligned}$$

This yields a statement more general than that of Theorem (4.3.1).

Theorem 4.4.1 *If g is defined as in (4.29), $a \in \mathbf{R}$, and using the notation of (4.30),*

$$\lim_{\substack{m\in J\\ m\to\infty}} \Pi_m g(2^{-m}a+b) = 2\int_{b_J}^{\infty} K_0(a+b_J, u)\, du - 1\,,$$

where $2^m b - [\![2^m b]\!] = b_J$ for $m \in J$.

Remark. If $b = 2^k$ for some integer k, then Theorem (4.4.1) simplifies to the case $b = 0$, which is Theorem (4.3.1).

4.4.2 Examples

To see if the dyadic sum Haar system expansion of g has a Gibbs effect at b, one needs to compute the integral

$$\int_{b_J}^{\infty} K_0(a+b_J,u)\,du = \int_{b_J}^{\infty} \sum_{n\in\mathbf{Z}} \phi(a+b_J-n)\phi(u-n)\,du.$$

Since $\phi(x) = \chi_{[0,1)}(x)$,

$$\begin{aligned}\int_{b_J}^{\infty} K_0(a+b_J,u)\,du &= \int_{b_J-[\![a+b_J]\!]}^{\infty} \chi_{[0,1)}(t)\,dt \\ &= \begin{cases} 1, & \text{if } [\![a+b_J]\!] \geq 1, \quad a \geq 1-b_J \\ 1-b_j, & \text{if } [\![a+b_J]\!] = 0, \quad -b_J \leq a < 1-b_J \\ 0, & \text{if } [\![a+b_J]\!] \leq -1, \quad a < -b_J. \end{cases}\end{aligned}$$

By the definition of b_J, $0 \leq b_J < 1$. Thus, by the above and Theorem (4.4.1),

$$\lim_{\substack{m\in J\\ m\to\infty}} \Pi_m g(2^{-m}a+b) = \begin{cases} 1, & \text{if } a \geq 1-b_J > 0 \\ 1-2b_j, & \text{if } -b_J \leq a < 1-b_J \\ -1, & \text{if } a < -b_J \leq 0, \end{cases}$$

and there is no Gibbs phenomenon in this case.

The next examples we look at are the compactly supported wavelets. Again, as done previously, computer analysis will be used to approximate values of necessary integrals. To see what region a Gibbs effect could occur, the integral $\int_{b_J}^{\infty} K_0(a+b_J,u)\,du = \int_{b_J}^{\infty} \sum_{n\in\mathbf{Z}}\, {}_N\phi(a+b_J+n)\,{}_N\phi(u+n)\,du$ needs to be examined. To begin with, for $a > 0$, when is

$$\int_{b_J}^{\infty} K_0(a+b_J,u)\,du \;\; = 1 \qquad \left(= \int_{\mathbf{R}} K_0(a+b_J,u)\,du\right) \tag{4.33}$$

true? Since,

$$\int_{b_J}^{\infty} K_0(a+b_J,u)\,du = \sum_{n=-\infty}^{2N-2} {}_N\phi(a+b_J+n) \int_{b_J+n}^{2N-1} {}_N\phi(t)\,dt \tag{4.34}$$

J	left side of origin		right side of the origin	
	a	$\lim_{m\to\infty} \Pi_m g(2^{-m}a+b)$	a	$\lim_{m\to\infty} \Pi_m g(2^{-m}a+b)$
even	-0.35	-1.15	0.65	1.32
odd	-1.70	-1.00 *	1.30	1.30
* *Computer calculated number smaller, but digits insignificant*				

Table 4.3: Approximate maximum overshoot and undershoot for dyadic sum wavelet expansions $\lim_{m\to\infty} \Pi_m g(2^{-m}a + b)$ generated by ${}_2\phi$, where $b = \frac{1}{3}$ (${}_2\phi$ is approximated by ${}_2\eta_{11}$).

and

$$\int_{\mathbf{R}} K_0(a+b_J,u)\,du = \sum_{n=-\infty}^{2N-1} {}_N\phi(a+b_J+n) \int_0^{2N-1} {}_N\phi(t)\,dt, \tag{4.35}$$

equation (4.33) is true when ${}_N\phi(a+b_J+n) = 0$ for $b_J+n > 0$. This occurs when $a+b_J+n \geq 2N-1$, or when $a \geq 2N-1$. Thus, by Theorem (4.4.1), no Gibbs effect will occur if $a \geq 2N-1$.

Similarly, if $a < 0$, when does the expression in equation (4.34) equal zero? This occurs when ${}_N\phi(a+b_J+n) = 0$ for $b_J+n < 2N-1$. This is the case if $a+b_J+n \leq 0$, or when $a \leq -(2N-1)$. Thus, in looking for a Gibbs phenomenon, the region that needs to be checked is $-(2N-1) < a < 2N-1$, where a is the number from Theorem (4.4.1).

Computer computations were done for the wavelet expansions of g generated by ${}_2\phi$, where the point of discontinuity is $b = 1/3$. In this case, $b_J = 1/3$ when $J = \{m : m \text{ is even}\}$, and $b_J = 2/3$ when $J = \{m : m \text{ is odd }\}$. The function ${}_2\phi$ was approximated by ${}_2\eta_{11}$, and the Gibbs phenomenon was checked for $-3 < a < 3$. Result are given in Table (4.3)

4.4.3 A general formula for partial sums

The last type of expansion that will be looked at is general partial sums of a function with a jump discontinuity at any point. This can be done by looking at partial sum of g. As in (2.23),

$$\mathcal{S}_m^{l\sigma} g(x) = \Pi_m g(x) + \int_{\mathbf{R}} g(y) G_m^{l\sigma}(x,y)\, dy, \tag{4.36}$$

where

$$G_m^{l\sigma}(x,y) = \sum_{k=0}^{l} \psi_{m,\sigma(k)}(x)\psi_{m,\sigma(k)}(y).$$

Similar to the work done for $\Pi_m g(2^m a + b)$, and using the notation of equation (4.32),

$$\begin{aligned}
\lim_{\substack{m\in J \\ m\to\infty}} \int_{\mathbf{R}} & g(y)\, G_m^{l\sigma}(2^{-m}a + b, y)\, dy \\
&= \int_{b_J}^{\infty} G_0^{l\sigma}(a + b_J, u)\, du - \int_{-\infty}^{b_J} G_0^{l\sigma}(a + b_J, u)\, du \\
&= 2\int_{b_J}^{\infty} G_0^{l\sigma}(a + b_J, u)\, du.
\end{aligned}$$

This gives us the following corollary.

Corollary 4.4.2 *For g defined in (4.29), $a, b \in \mathbf{R}$, and using the notation of equation (4.36),*

$$\lim_{\substack{m\in J \\ m\to\infty}} \mathcal{S}_m^{l\sigma} g(2^{-m}a + b) = \lim_{\substack{m\in J \\ m\to\infty}} \Pi_m g(2^{-m}a + b) + \mathcal{G}_{J,b}^{l,\sigma}(a),$$

where

$$\mathcal{G}_{J,b}^{l,\sigma}(a) \equiv 2\int_{b_J}^{\infty} G_0^{l\sigma}(a + b_J, u)\, du.$$

Applications of this result will be given in the next section.

4.4.4 Examples

In this last section, specific example of general partial sum expansions of a function with a jump discontinuity at a general point will be examined. This examination will be based on Corollary (4.4.2).

Once again, the Haar system will be studied first. In Section (4.4.2) the dyadic sum term of the expansion of g was found to be

$$\lim_{\substack{m\in J\\ m\to\infty}} \Pi_m g(2^{-m}a+b) = \begin{cases} 1, & \text{if } a \geq 1-b_J > 0 \\ 1-2b_J & \text{if } -b_J \leq a < 1-b_J \\ -1 & \text{if } a < -b_J \leq 0 \end{cases} \tag{4.37}$$

This leaves the term $\mathcal{G}^{l,\sigma}_{J,b}(a)$ to be computed.

$$\begin{aligned} \mathcal{G}^{l,\sigma}_{J,b}(a) &= 2\int_{b_J}^{\infty} G_0^{l\sigma}(a+b_J,u)\,du \\ &= 2\sum_{k=0}^{l} \psi(a+b_J-\sigma(k)) \int_{b_J}^{\infty} \psi(u-\sigma(k))\,du \end{aligned}$$

For the Haar system, $\psi(t) = \chi_{[0,\frac{1}{2})}(t) - \chi_{[\frac{1}{2},1)}(t)$, and by the definition of b_J,(4.32), $0 \leq b_J < 1$. Thus, the above equation simplifies to

$$\mathcal{G}^{l,\sigma}_{J,b}(a) = \begin{cases} 2\psi(a+b_J)\int_{b_J}^{1}\psi(t)\,dt, & \text{if } \sigma(k)=0 \text{ for some } 0 \leq k \leq l \\ 0, & \text{otherwise.} \end{cases}$$

This first term can only be nonzero if $-b_J < a < 1-b_J$. Values for this region will be given in Table (4.4), and values for the limit of the total partial sum expansion will be given using (4.37) and Corollary (4.4.2).

It has now been verified that there is no Gibbs effect for the general partial sum expansion of a function with a jump discontinuity at any point. This shows that in all cases concerning the Haar system, no Gibbs phenomenon occurs.

The last example that will be looked at in this section is a general partial sum of g generated by ${}_2\phi$. Again, we will use computer analysis.

$a+b_J$	b_J	$\mathcal{G}^{l,\sigma}_{J,b}(a)$	$\lim_{\substack{m\in J\\ m\to\infty}} \mathcal{S}^{l\sigma}_m g(2^{-m}a+b)$
$0\le a+b_J<\frac{1}{2}$	$0\le b_J<\frac{1}{2}$	$-2b_J$	$1-4b_J\in[-1,1]$
$0\le a+b_J<\frac{1}{2}$	$\frac{1}{2}\le b_J<1$	$-2+2b_J$	-1
$\frac{1}{2}\le a+b_J<1$	$0\le b_J<\frac{1}{2}$	$2b_J$	1
$\frac{1}{2}\le a+b_J<1$	$\frac{1}{2}\le b_J<1$	$2-2b_J$	$3-4b_J\in[-1,1]$

Table 4.4: Values of $\lim_{\substack{m\in J\\ m\to\infty}} \mathcal{S}^{l\sigma}_m g(2^{-m}a+b)$ for the Haar system in the region where $\mathcal{G}^{l,\sigma}_{J,b}(a)\neq 0$.

Before this can be done, it is necessary to determine what range of a and $\sigma(n)$ can give nonzero values for the term $\mathcal{G}^{l,\sigma}_{J,b}(a)$ of Corollary (4.4.2). This will be done for the more general case of ${}_N\phi$, a general compactly supported wavelet.

$$\begin{aligned}\mathcal{G}^{l,\sigma}_{J,b}(a) &= 2\int_{b_J}^{\infty} G^{l\sigma}_0(a+b_J,u)\,du \\ &= 2\sum_{k=0}^{l} {}_N\psi(a+b_J+\sigma(k))\int_{b_J+\sigma(k)}^{\infty} {}_N\psi(t)\,dt.\end{aligned}$$

Since $\int_{\mathbf{R}} {}_N\psi(t)\,dt = 0$, $\operatorname{supp}{}_N\psi = [-\frac{N}{2},N]$ and $0\le b_J<1$, $\mathcal{G}^{l,\sigma}_{J,b}(a)\neq 0$ implies that

$$-\frac{N}{2}-b_J<\sigma(k)<N-b_J$$

and

$$-N-\frac{N}{2}<a<N+\frac{N}{2}.$$

For the specific case of ${}_2\phi$, nonzero $\mathcal{G}^{l,\sigma}_{J,b}(a)$ terms may occur for $\sigma(k)=-1,0$ and 1 and $-3<a<3$. This area for a is the same area that was checked for a Gibbs phenomenon in the dyadic sum case. The computer program for this part estimated the $\mathcal{G}^{l,\sigma}_{J,b}(a)$ term for $-3<a<3$, $l=0,1,2$ or 3, $\sigma(n)$ taking any one of the values $-1,0$, and 1, and J being the set of odd or even values of m. As with the dyadic case, b will be chosen to be $1/3$. The results of the computer estimates are given in Tables (4.5) and (4.6).

This data seems to suggest that the size and the location of the maximum overshoot and undershoot may vary with the addition of extra terms in a general partial sum. A few observations from the data in the tables will be noted below.

In Section 4.3.4, the addition of extra terms to the dyadic sum seemed to reduce the Gibbs effect. The results in this section seem to show that the addition of terms can also increase the Gibbs phenomenon. Since the changes in the overshoot and undershoot are small when considering possible errors in the estimated values, more analysis in needed to study these results. Thus far, we have been limited by the computer time necessary to receive better estimates.

Another point of interest occurs when $J = \{m : m \text{ odd }\}$. The value for a for greatest undershoot appears to change. Since the changes in the undershoot may be insignificant because of estimation accuracy, more study is needed in the area. This would require faster computer algorithms or faster computers.

This chapter only begins to address questions concerning Gibbs phenomenon for wavelet expansions. As mentions above, faster computers and more efficient programs may allow more accurate data in the results given. These improvements would also allow one to study compactly supported wavelets with larger support sizes and look for possible patterns related to support size and overshoot and undershoot size. It would also be interesting to look at Gibbs effect for other types of wavelets, both with mathematical formulas and with the aid of the computers. This author hopes to be able to address some of these points in the future.

This chapter has illustrated a test for the Gibbs phenomenon which consists of computing the size of wavelet kernels over certain intervals. Exact computations were done for the Haar system, and computer estimates were done for some compactly supported wavelets.

J	l	$\sigma(n)$	a	$\lim \mathcal{S}_m^{l\sigma} g(2^{-m}a+b)$*	a	$\lim \mathcal{S}_m^{l\sigma} g(2^{-m}a+b)$*
even	0		-0.34	-1.15	0.66	1.34
even	1	$\sigma(1)=-1$	-0.34	-1.15	0.66	1.32
even	1	$\sigma(1)=0$	-0.34	-1.15	0.66	1.34
even	1	$\sigma(1)=1$	-0.34	-1.19	0.66	1.34
even	2	$\sigma(1)=-1$ $\sigma(2)=0$	-0.34	-1.15	0.66	1.33
even	2	$\sigma(1)=-1$ $\sigma(2)=1$	-0.34	-1.19	0.66	1.32
even	2	$\sigma(1)=0$ $\sigma(2)=1$	-0.34	-1.18	0.66	1.34
even	3	$\sigma(1)=-1$ $\sigma(2)=0$ $\sigma(3)=1$	-0.34	-1.18	0.66	1.33
* *stands for* $\lim_{\substack{m\in J \\ m\to\infty}} \mathcal{S}_m^{l\sigma} g(2^{-m}a+b)$						

Table 4.5: Approximate maximum overshoot and undershoot for general partial wavelet sums $\lim_{\substack{m\in J \\ m\to\infty}} \mathcal{S}_m^{l\sigma} g(2^{-m}a+b)$ generated by ${}_2\phi$, where $J=\{m : m \text{ even }\}$ and $b=\frac{1}{3}$. (${}_2\phi$ is approximated by ${}_2\eta_{11}$).

J	l	$\sigma(n)$	a	$\lim \mathcal{S}_m^{l\sigma} g(2^{-m}a+b)^*$	a	$\lim \mathcal{S}_m^{l\sigma} g(2^{-m}a+b)^*$
odd	0		-1.67	-1.002**	1.33	1.33
odd	1	$\sigma(1)=-1$	-1.67	-1.002**	1.33	1.16
odd	1	$\sigma(1)=0$	-1.67	-1.002**	1.33	1.33
odd	1	$\sigma(1)=1$	-1.17	-1.004**	1.33	1.33
odd	2	$\sigma(1)=-1$ $\sigma(2)=0$	-1.67	-1.002**	1.30	1.16
odd	2	$\sigma(1)=-1$ $\sigma(2)=1$	-1.17	-1.004**	1.33	1.16
odd	2	$\sigma(1)=0$ $\sigma(2)=1$	-1.67	-1.001**	1.33	1.33
odd	3	$\sigma(1)=-1$ $\sigma(2)=0$ $\sigma(3)=1$	-1.67	-1.001**	1.33	1.16
* *stands for* $\lim_{\substack{m\in J\\ m\to\infty}} \mathcal{S}_m^{l\sigma} g(2^{-m}a+b)$						
** *last digits may be insignificant.*						

Table 4.6: Approximate maximum overshoot and undershoot for general partial wavelet sums $\lim_{\substack{m\in J\\ m\to\infty}} \mathcal{S}_m^{l\sigma} g(2^{-m}a+b)$ generated by ${}_2\phi$, where $J=\{m : m \text{ odd }\}$ and $b=\frac{1}{3}$. (${}_2\phi$ is approximated by ${}_2\eta_{11}$).

Computational analysis for Chapter 4

In this appendix, we will give a discussion of the FORTRAN programs used to estimate the Gibbs effect for expansions by compactly supported wavelets of a function with a jump discontinuity. The data from these programs will also be analyzed. In all of this appendix, we shall be using the compactly supported wavelets of I. Daubechies, and as in Chapter 4, we will simplify notation by using ϕ in place of ${}_N\phi$.

Computer programs were used to estimate the Gibbs phenomenon in four cases: **(1)** in a dyadic sum expansions of a function with a jump discontinuity at the origin, **(2)** in a general partial sum expansion of a function with a jump discontinuity at the origin, **(3)** in a dyadic sum expansion of a function with a jump discontinuity at 1/3, and **(4)** in a general partial sum expansion of a function with a jump discontinuity at 1/3 (see Sections 4.3.2, 4.3.4, 4.4.2 and 4.4.4).

Let us examine Case 1, Gibbs phenomenon for dyadic sum expansions of a function with a jump discontinuity at the origin; the function, $f(x)$, used is found in equation (4.13). From Theorem (4.3.1), we have that $\lim_{m\to\infty} \Pi_m f(2^{-m}a) = 2\int_0^\infty K_0(a,u)\,du - 1$, and from the work in Section 4.3.2, for a dyadic wavelet expansion generated by ϕ, a Gibbs effect can only occur for values of a between

$-(2N-2)$ and $2N-2$. With these results, estimating the size of a Gibbs effect is quite simple. The computer is used find a value for the integral $\int_0^\infty K_0(a,u)\,du = \sum_{n=-\infty}^{2N-2} \phi(a+n) \int_n^{2N-1} \phi(t)\,dt$. This value can only be estimated since we have no closed form for ϕ. Using equation (4.9), we will approximate ϕ by values of η_l.

To simplify our discussion, let us look at the specific case where $N=2$ and $a>0$. In this case, we wish to evaluate

$$\int_0^\infty K_0(a,u)\,du = \sum_{n=-2}^{2} \eta_l(a+n) \int_n^3 \eta_l(t)\,dt\,. \tag{A.1}$$

This equation can be evaluated exactly, except for computer round off errors, since η_l is a simple step function whose value at any step is determined iteratively with equation (4.10); here $h(0) = (1+\sqrt{3})/4\sqrt{2}$, $h(1) = (3+\sqrt{3})/4\sqrt{2}$, $h(2) = (3-\sqrt{3})4\sqrt{2}$, and $h(3) = (1-\sqrt{3})/4\sqrt{2})$ (see [Da1]). The computer programs evaluated equation (A.1) for values of a between 0 and 2 in increments of 0.01, and for l values of 3 through 11. Similar work was done in the case where a is negative. The function η_{11} was the last estimating function used because of the large amount of time necessary to compute values for these functions iteratively. Data and analysis for this case are given in Tables (A.1) and (A.2).

The following technique has been used to analyze the data in the above tables. Let

$$\Theta_\phi = \text{overshoot (undershoot) with expansion generated by } \phi,$$

$$\Theta_{\eta_j} = \text{overshoot (undershoot) with expansion generated by } \eta_j,$$

$$d_j = \Theta_\phi - \Theta_{\eta_j},$$

and

$$r = \text{approximate ratio of } d_j \text{ and } d_{j+1}.$$

In this work, we will assume that the above ratio is independent of j. Since the data in the tables suggest that the ratio is actually decreasing, we will actually be

j	$\lim_{m\to\infty}\Pi_m f(2^{-m}a)$ $(2\int_0^\infty K_0(a,u)\,du-1)$	Overshoot from 1	Difference of Overshoot	Ratio of Difference
3	1.201484870	0.201484870		
			0.120487212	
4	1.321972082	0.321972082		0.720544152
			0.0868163536	
5	1.408788438	0.408788438		0.717141572
			0.062259618	
6	1.471048056	0.471048056		0.716942818
			0.044636586	
7	1.515684642	0.515684642		0.713948688
			0.031868232	
8	1.547552874	0.547552874		0.709062304
			0.022596562	
9	1.570149436	0.570149436		0.703819811
			0.015903908	
10	1.586053344	0.586053344		0.699093204
			0.011118314	
11	1.597171658	0.59717658		

Table A.1: Computer data for estimating the Gibbs effect on the right hand side of the origin using dyadic wavelet expansions generated by ${}_2\phi$. The value of a in this data is 1.0, which produced the maximum overshoot. The function ${}_2\phi$ is approximated by η_j.

j	$\lim_{m\to\infty} \Pi_m f(2^{-m}a)$ $(2\int_0^\infty K_0(a,u)\,du - 1)$	Undershoot from -1	Difference of Undershoot	Ratio of Difference
3	-1.00742503	-0.00742503		
			-0.011869768	
4	-1.019294798	-0.019294798		0.799229943
			-0.009486674	
5	-1.028781472	-0.028781472		0.600112326
			-0.005693070	
6	-1.034474542	-0.034474542		0.538555121
			-0.003066032	
7	-1.037540574	-0.037540574		0.58964290
			-0.001807864	
8	-1.039348438	-0.039348438		0.566461858
			-0.001024086	
9	-1.040372524	-0.040372524		0.492972270
			-0.000504846	
10	-1.040877370	-0.040877370		0.485157850
			-0.000244930	
11	-1.041122230	-0.041122230		

Table A.2: Computer data for estimating the Gibbs effect on the left hand side of the origin using dyadic wavelet expansions generated by ${}_2\phi$. The value of a in this data is -1.1, which produced the maximum undershoot. The function ${}_2\phi$ is approximated by η_j.

computing estimates for the upper bounds of the overshoot and undershoot. Now,

$$\begin{aligned}\Theta_\phi &= \lim_{j\to\infty}\Theta_{\eta_j} \\ &= \Theta_{\eta_j} + d_j + d_{j+1} + d_{j+2} + ... \\ &\approx \Theta_{\eta_j} + d_j\sum_{j=0}^{\infty} r^j \\ &= \Theta_{\eta_j} + d_j/(1-r). \end{aligned} \tag{A.2}$$

Using the data from Tables (A.1) and (A.2) in equation (A.2), we will let $j = 10$ and r be the smallest ratio in each table. The computations yield $\Theta_\phi \approx 0.62300$ on the right hand side of the origin, and $\Theta_\phi \approx -0.04135$ on the left hand side of the origin. These values are quite close to the values of the overshoot and undershoot computed using η_{11}, so our estimates look reasonable.

Similar programs and computations were done for the expansions generated by ${}_3\phi$, ${}_4\phi$, and ${}_5\phi$. Less data was collected in these cases since more time was required for the iterative process in these cases.

This finishes the explanation of the programs and analysis used in Case 1. In the rest of the cases, only expansion by ${}_2\phi$ and ${}_2\psi$ were used. In these cases, ${}_2\phi$ is approximated by η_{11}, and ${}_2\psi$ is computed using the equation

$${}_2\psi(x) = \sum_{n=-2}^{1}(-1)^n h(-n+1)\,{}_2\phi(2x-n).$$

These programs follow directly from the discussions in Sections 4.3.4, 4.4.2 and 4.4.4. This completes the discussion of the computer programs for Chapter 4.

Bibliography

[Au-We-Wi1] P. Auscher, G. Weiss and V. Wickerhauser, *Local sine and cosine bases of Coifman and Meyer and the construction of smooth wavelets* to be published.

[Ba] G. Battle, *A block spin construction of ondelettes. Part I: Lemarié functions*, Commun. Math. Phys. **110** (1987), 601-615.

[Co-Me] R.R. Coifman and Y. Meyer, *Remarques sur l'analyse de Fourier à fenêtre*,série I, C.R. Acad. Sci. Paris, **312** (1991), 259-261.

[Co-Ro] R.R. Coifman and R. Rochberg, *Representation theorems for holomorphic and harmonic functions in L^p*, Astérisque **77** (1980), 11-66.

[Co-We] R.R. Coifman and G. Weiss, *Extensions of Hardy spaces and their uses in analysis*, Bull. Amer. Math. Soc., **83** (1977), 569-645.

[Da1] I. Daubechies, *Orthonormal bases of compactly supported wavelets*, Comm. on Pure and Appl. Math., **XLI** (1988), 909-996.

[Da2] I. Daubechies, *The wavelet transform: a method for time- frequency localization*, preprint.

[Fo-Ri] J. Foster and F.B. Richards, *A Gibbs phenomenon for piecewise-linear approximation*, The Am. Math. Month. **Jan.** (1991), 47-49.

[Fr-Ja1] M. Frazier and B. Jawerth, *Decomposition of Besov spaces*, Indiana Univ. Math. J. **34** (1985), 777-799.

[Fr-Ja2] M. Frazier and B. Jawerth, *The φ-transform and applications to distribution spaces*, Function Spaces and Applications, M. Cwikel et al., eds, Springer Lecture Notes in Math. **1302** (1988), 233-246.

[Fr-Ja3] M.Frazier and B. Jawerth, *A discrete transform and decomposition of distribution spaces*, to appear in J. Func. Anal.

[Fr-Ja-We] M. Frazier, B. Jawerth, and G. Weiss, *Littlewood -Paley theory and the study of function spaces*, to appear as a monograph in the CBMS-AMS Regional Conference Series.

[Gi] J.W. Gibbs, *letter to the editor*, Nature **59** (1899), 606.

[Ja] S. Jaffard, *Pointwise regularity of functions and wavelet coefficients*, Chapter 4 of Ph.D. thesis.

[Le] P.G. Lemarié, *Ondelettes á localisation exponentielle*, to appear in J. Math. Pures et Appl.

[Le-Me] P.G. Lemarié and Y. Meyer, *Ondelettes et bases hilbertennes*, Rev. Math Ibervamericana **2** (1986), 1-18.

[Ma] S. Mallat, *Multiresolution approximations and wavelet orthonormal bases of* $L^2(\mathbf{R})$, Trans. of the Am. Math. Soc., **315** (1989), no. 1, 69-87.

[Me] Y. Meyer, *Wavelets and operators*, notes from lectures given at the University of Illinois, (1986)

[Mi] A.A. Michelson, *letter to the editor*, Nature **58** (1898), 544-45.

[Ri] F.B. Richards, *A Gibbs phenomenon for spline functions*, to appear in J. Approx. Theory.

[So] J. Soria, private communication.

[St] J.-O. Stömberg, *A modified Franklin system and higher- order spline system on* $\mathbf{R}^n$ *as unconditional bases for Hardy spaces*, Conference on Harmonic Analysis in honor of Antoni Zygmund, Vol. II, W. Beckner et al., eds, Wadsworth Math. Series. (1981), 475-494.

[Ta-We] M. Taibleson and G. Weiss, *The molecular characterization of certain Hardy spaces*, Astérisque, **77** (1980), 67-149.

[We] G. Weiss, *Franklin wavelets and spline wavelets (the periodic case)*, lecture notes from a course given at the University of California at Berkley, (1988).

[Zy] A. Zygmund, *Trigonometric Series*, Cambridge Mathematical Library, 2nd ed. (1959).