Project 1: Due Thursday Oct. 29nd

Markov Chains and the Growth of Trees

IMPORTANT Complete the following problems on a separate sheet of paper. Show all your work and please answer the questions in complete sentences. In addition to grading the mathematical correctness of the project, I am also going to grade the form, content and presentation of the project (see the writing guidelines). In the following project, the questions are provided to direct your investigation. Your report should be a stand-alone document, readable without this project description.

Part I - Introduction and Set-up

Many physical situations that involve changing quantities can be modeled with Calculus. Often it is difficult (if not impossible) to know precise information about variables required by calculus-based models. Markov Chains is a method of modeling future values based on the probability they will move from their current state. Section 11.6 in the textbook provides a good introduction to Markov chains including a variety of application examples.

Read section 11.6 on pages 608-619 in the book before you begin this project.

Suppose we are interested in a forest that is composed of two species of trees, with A_t and B_t denoting the number of each species in the forest in year t. When a tree dies, a new tree grows in its place, but the new tree might be either species.

The weather has a large impact on the groth of each species. The rejuvenation and growth of the forest depends on the weather for that particular year. Many forests near the equator experience wet seasons and dry seasons. In a wet season, the accumulated rainfall for that year is much larger than during a dry season.

A wet season.

Assume the species A tree are relatively long lived, with only 1% dying in any given year. On the other hand, the growth of the species B tree depends on the type of season. In a wet season, 5% of the species B trees die. Because they are rapid growers, the B trees, however, are more likely to succeed in winning a vacancy spot left by a dead tree; 75% of all vacant spots go to species B trees, and only 25% go to species A trees.

Let $\vec{x}^{(t)} = \begin{bmatrix} A_t \\ B_t \end{bmatrix}$ be the state vector for this model. The vector represents the number of each species of tree after t years. The observations above describe how the numbers of each species in the forest transitions/changes into the values for the following year. This can be express by the equations

$$\begin{aligned} A_{t+1} &= (0.99 + (0.25)(0.01))A_t + (0.25)(0.05)B_t \\ B_{t+1} &= (0.75)(0.01)A_t + (0.95 + (0.75)(0.05))B_t \end{aligned}$$

After simplifying, the model can be reduced to the matrix equation

$$\vec{x}^{(t+1)} = W\vec{x}^{(t)} \qquad \text{where} \quad W = \begin{bmatrix} 0.9925 & 0.0125 \\ 0.0075 & 0.9875 \end{bmatrix} \tag{1}$$

The future number of trees $\vec{x}^{(t+1)} = \begin{bmatrix} A_{t+1} \\ B_{t+1} \end{bmatrix}$ is found by multiplying the present number $\vec{x}^{(t)}$ by the transition matrix W.

• Suppose we begin with populations $A_0 = 10$ and $B_0 = 990$. These initial population values might describe the forest if most of the A trees were selectively logged in the past. Derive an equation to find the populations after k wet years. Find the steady state vector for equation (**??**). Discuss its meaning in the context of this problem.

A dry season.

During dry years, species B dies at a greater rate than species A. Suppose that the likelihood of a B tree dying in a dry year is now 0.39.

• Derive a model (similar to equation (**??**)) for a dry year. Explain the source and meaning of each term in your transition matrix.

• Suppose that a dry year is followed by a wet year; how should the populations change? Find a single transition matrix for this situation. Would you expect the effect on a forest from dry year followed by a wet year to be exactly the same as that of a wet year followed by a dry year? Explain.

• Suppose that D is the projection matrix for a dry year. If the first year is dry and the second is wet, explain how you can find the initial population $\vec{x}^{(0)}$ knowing only the value two years later $\vec{x}^{(2)}$. Use your argument to show that $(WD)^{-1} = D^{-1}W^{-1}$.

Writing Guidelines

- **Writing Style:** Proper mathematical prose integrates text, derivations, calculations, diagrams, and explanations seamlessly using complete sentences with proper grammar. Use your textbook as a guide.

- **Amount of Detail:** As a general rule, you should put in just enough detail so that a fellow linear algebra student can follow your explanation. You want to provide some insight as to where things came from without making the reader pour over lots of algebra or trivial calculations.

- **Collaboration:** You are encouraged in pairs and submit a single copy for grading. You may, however, choose to work individually. *Please attach a short summary indicating the contributions from each partner in the project.*

- **General Grading Criteria:** A general scoring rubric will be used to evaluate your writing projects. Each part will be assigned a certain percentage of the total points. In particular, Mathematical Correctness will represent at least 50 percent of the points.

- **Questions:** I expect and welcome questions about the mathematics, the level of detail required, and how to write up / typeset your results. Feel free to email, stop, or make an appointment to get questions answered.