Energy Balance for Dynamic Contact with Signorini's Condition and Slip Rate Dependent Friction

Ted Wendt

Department of Mathematics
The University of Iowa
(This research was funded by The University of Iowa Department of Mathematics NSF VIGRE grant DMS-0602242.)

Southeast Atlantic Regional Conference on Differential Equations
October 20, 2007

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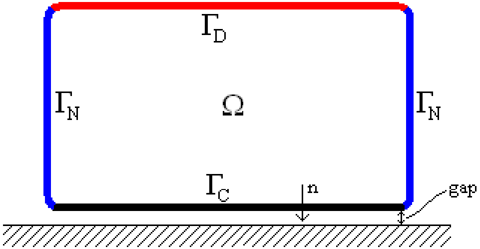
The Problem

In this talk we consider the dynamic frictional contact between a viscoelastic body and a rigid foundation. Let Ω represent the body and Γ be its boundary. Partition Γ into three disjoint parts: the contact boundary Γ_C, the fixed boundary Γ_D, and the Neumann boundary Γ_N.



Classical Formulation

Let $\mathbf{u}(\mathbf{x}, t)$ represent the displacement of the point $\mathbf{x}$ at time t, and set $\mathbf{v} = \mathbf{u}'$. The classical formulation for this problem is

$$\begin{aligned} \mathbf{v}' &= \mathrm{div}\,(\sigma(\mathbf{u},\mathbf{v})) + \mathbf{f}_B \qquad \mathrm{in}\,\Omega_T = \Omega \times (0, T) \\ \sigma(\mathbf{u},\mathbf{v}) &= A\varepsilon(\mathbf{u}) + B\varepsilon(\mathbf{v}) \\ \varepsilon(\mathbf{u}) &= \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right) \end{aligned}$$

Boundary Conditions

We use Dirichlet conditions on Γ_D, Neumann conditions on Γ_N, and Signorini complementarity conditions on Γ_C.

$$\mathbf{u}(\mathbf{x},t) = 0 \qquad \text{on}\,\Gamma_D$$
$$\sigma \cdot \mathbf{n}(\mathbf{x},t) = \mathbf{f}_N \qquad \text{on}\,\Gamma_N$$
$$0 \leq g - u_n \perp -\sigma_n \geq 0 \qquad \text{on}\,\Gamma_C$$

Handling Friction

In [1], Kuttler and Shillor use a regularized nonlocal contact stress in order to avoid defining the trace of the stress on the contact boundary. They use this regularization in the context of the generalized Coulomb law for sliding friction:

$$
\begin{aligned}
|\sigma_T| \leq \mu(|\mathbf{v}_T|)|(\mathscr{R}\sigma)_n| \qquad & \text{on}\,\Gamma_C \\
\text{if}\,\mathbf{v}_T \neq 0, \text{then}\,\sigma_T = -\mu(|\mathbf{v}_T|)|(\mathscr{R}\sigma)_n|\frac{\mathbf{v}_T}{|\mathbf{v}_T|}&.
\end{aligned}
$$

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Spaces

Using the notation described in the previous section, we choose spaces U, V, H, $\mathscr{U}$, $\mathscr{V}$, $\mathscr{H}$, and $\mathscr{K}$ as follows:

$$V = \left\{ \mathbf{u} \in (H^1(\Omega))^N : \mathbf{u} = 0 \,\text{on}\, \Gamma_D \right\}.$$

Set $H = \left(L^2(\Omega)\right)^N$, so that

$$V \subseteq H = H' \subseteq V'$$

Let U be a Banach space in which V is compactly embedded, V is dense in U, and the trace map $\gamma : U \to (L^2(\partial\Omega))^N$ is continuous. (e.g. $U = (H^{3/4}(\Omega))^N \cap V$).

Spaces

- $\mathscr{V} = L^2(0, T; V)$
- $\mathscr{H} = L^2(0, T; H)$
- $\mathscr{K} = \left\{ \mathbf{w} \in \mathscr{V} : \mathbf{w}' \in \mathscr{V}, w_n - g \leq 0 \, \text{in} \, L^2(0, T; L^2(\Gamma_C)) \right\}$
- $\mathscr{K}_{\mathbf{u}} = \{ \mathbf{w} \in \mathscr{K} : \mathbf{w}(T) = \mathbf{u}(T) \}$.

More Notation

Define operators $M, L : V \to V'$ by

$$\begin{aligned} \langle M\mathbf{u}, \mathbf{v}\rangle &= \int_\Omega (B\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))\, dx \\ \langle L\mathbf{u}, \mathbf{v}\rangle &= \int_\Omega (A\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))\, dx. \end{aligned}$$

Define $\mathbf{f} \in L^2(0, T; V')$ by

$$\langle \mathbf{f}, \mathbf{z}\rangle_{\mathscr{V}', \mathscr{V}} = \int_0^T \int_\Omega \mathbf{f}_B \cdot \mathbf{z} d\mathbf{x}\, dt + \int_0^T \int_{\Gamma_N} \mathbf{f}_N \cdot \mathbf{z} d\Gamma dt$$

for $\mathbf{z} \in \mathscr{V}$. Finally, define the adjoint trace operator $\gamma_T^* : L^2(0, T; (L^2(\Gamma_C))^N) \to \mathscr{V}'$ by

$$\langle \gamma_T^* \xi, \mathbf{w}\rangle = \int_0^T \int_{\Gamma_C} \xi \cdot \mathbf{w}_T d\Gamma dt.$$

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Weak Formulation

The classical formulation is equivalent to: Find

- $\mathbf{u} \in C([0,T];U) \cap L^\infty(0,T;V)$
- $\mathbf{v} \in L^2(0,T;V) \cap L^\infty(0,T;H)$
- $\mathbf{v}' \in L^2(0,T;H^{-1}(\Omega)^N)$
- $\xi \in L^2(0,T;L^2(\Gamma_C)^N)$

such that for all $\mathbf{w} \in \mathscr{K}_{\mathbf{u}}$

$$-(\mathbf{v}(0),\mathbf{u}(0)-\mathbf{w}(0))_H - \int_0^T (\mathbf{v},\mathbf{v}-\mathbf{w}')_H \, dt + \int_0^T \langle M\mathbf{v},\mathbf{u}-\mathbf{w}\rangle \, dt + \int_0^T \langle L\mathbf{u},\mathbf{u}-\mathbf{w}\rangle \, dt - \langle \gamma_T^* \xi, \mathbf{u}-\mathbf{w}\rangle \leq \langle \mathbf{f},\mathbf{u}-\mathbf{w}\rangle_{\mathscr{V}',\mathscr{V}}$$

where $\gamma_T^* \xi$ satisfies for $\mathbf{w} \in \mathscr{K}_{\mathbf{u}}$,

$$\langle \gamma_T^* \xi, \mathbf{w}\rangle \leq \int_0^T \int_{\Gamma_C} \mu(|\mathbf{v}_T|)|(\mathscr{R}\sigma)_n|(|\mathbf{v}_T - \mathbf{w}_T| - |\mathbf{v}_T|) d\Gamma dt.$$

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Stewart's Lemma

In *Energy Balance for Viscoelastic Bodies in Frictionless Contact*, Stewart proved the following lemma:

Lemma

Let K be a closed, convex set in a reflexive Banach space X, $f : [0,T] \to X'$, $f \in L^p(0,T;X')$, and $u : [0,T] \to X$ be a solution to the variational inequality: $u(t) \in K$ for all t, and

$$\int_0^T \langle \tilde{u}(t) - u(t), f(t) \rangle_{X \times X'} \, dt \geq 0 \qquad \text{for all continuous } \tilde{u} : [0,T] \to K.$$

Then provided the distributional derivative $u' \in L^q(0,T;X)$ $(1/p + 1/q = 1)$, then

$$\left\langle u'(t), f(t) \right\rangle_{X \times X'} = 0 \qquad \text{for almost all } t \in [0,T].$$

Sketch of Proof

- Since $\int_0^T \langle \tilde{u}(t) - u(t), f(t) \rangle \, dt \geq 0$ for all continuous $\tilde{u}$, we can show that $\langle w - u(t), f(t) \rangle \geq 0$ for all $w \in K$ and almost all t.
- The set $E \subset [0, T]$ on which $\langle w - u(t), f(t) \rangle < 0$ is a null set.
- The ordinary derivative $\lim_{h \to 0} (u(t+h) - u(t))/h$ exists almost everywhere, except on a null set, F.
- On $[0, T] \setminus (E \cup F)$,

$$\langle u'(t), f(t) \rangle = \lim_{h \to 0} \left\langle \frac{u(t+h) - u(t)}{h}, f(t) \right\rangle$$

- As $h \to 0$, this limit is 0.

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- As $h \to 0$, this limit is 0.

Outline

Properties of $\mathscr{K}_u$

Recall that we defined

- $\mathscr{K} = \left\{\mathbf{w} \in \mathscr{V} : \mathbf{w}' \in \mathscr{V}, w_n - g \leq 0 \,\mathrm{in}\, L^2(0, T; L^2(\Gamma_C))\right\}$
- $\mathscr{K}_{\mathbf{u}} = \{\mathbf{w} \in \mathscr{K} : \mathbf{w}(T) = \mathbf{u}(T)\}$

Claim

$\mathscr{K}_{\mathbf{u}}$ is closed and convex in $\mathscr{V} \subseteq \mathscr{H} = \mathscr{H}'$.

Applying Stewart's Lemma

Put $\mathbf{F} = \mathbf{v}' + M\mathbf{v} + L\mathbf{u} + \gamma_T^* \xi - \mathbf{f}$, so that $\langle \mathbf{F}, \mathbf{w} \rangle = \left\langle \mathbf{v}' + M\mathbf{v} + L\mathbf{u} + \gamma_T^* \xi - \mathbf{f}, \mathbf{w} \right\rangle_{V \times V'}$ for all $\mathbf{w}$, and suppose that $\mathbf{u}$ is a weak solution to the problem. Then

$$\int_0^T \left\langle \mathbf{v}' + M\mathbf{v} + L\mathbf{u} + \gamma_T^* \xi - \mathbf{v}, \mathbf{u} - \mathbf{w} \right\rangle_{V \times V'} dt \leq 0 \qquad \text{for all } \mathbf{w} \in \mathscr{K}_{\mathbf{u}}$$

or equivalently,

$$\int_0^T \langle \mathbf{w} - \mathbf{u}, \mathbf{F} \rangle_{V' \times V} \, dt \geq 0 \qquad \text{for all } \mathbf{w} \in \mathscr{K}_{\mathbf{u}}.$$

Since $\mathscr{K}_{\mathbf{u}}$ is closed and convex, and $\mathbf{u}' = \mathbf{v} \in L^2(0, T; V)$, we conclude that

$$\langle \mathbf{v}, \mathbf{F} \rangle_{V' \times V} = 0 \text{ for a.e. } t \in [0, T].$$

Applying Stewart's Lemma

That is,

$$\begin{aligned} 0 = &\left\langle \mathbf{v}(t), \mathbf{v}'(t) \right\rangle + \langle L\mathbf{u}(t), \mathbf{v}(t) \rangle + \langle M\mathbf{v}(t), \mathbf{v}(t) \rangle \\ &+ \langle \gamma_T^* \xi(t), \mathbf{v}(t) \rangle - \langle \mathbf{f}(t), \mathbf{v}(t) \rangle \\ \left\langle \mathbf{v}(t), \mathbf{v}'(t) \right\rangle + \langle L\mathbf{u}(t), \mathbf{v}(t) \rangle &= \langle \mathbf{f}(t), \mathbf{v}(t) \rangle - \langle M\mathbf{v}(t), \mathbf{v}(t) \rangle - \langle \gamma_T^* \xi(t), \mathbf{v}(t) \rangle \end{aligned}$$

Observe that

$$\left\langle \mathbf{v}(t), \mathbf{v}'(t) \right\rangle = \frac{1}{2} \frac{d}{dt} \langle \mathbf{v}(t), \mathbf{v}(t) \rangle$$

and

$$\langle L\mathbf{u}(t), \mathbf{v}(t) \rangle = \frac{1}{2} \frac{d}{dt} \langle L\mathbf{u}(t), \mathbf{u}(t) \rangle .$$

Applying Stewart's Lemma

Therefore,

$$\frac{d}{dt}\left(\frac{1}{2}\langle \mathbf{v}(t), \mathbf{v}(t)\rangle + \frac{1}{2}\langle L\mathbf{u}(t), \mathbf{u}(t)\rangle\right) = \langle \mathbf{f}(t), \mathbf{v}(t)\rangle - \langle M\mathbf{v}(t), \mathbf{v}(t)\rangle - \langle \gamma_T^* \xi(t), \mathbf{v}(t)\rangle .$$

- $\frac{1}{2}\langle \mathbf{v}(t), \mathbf{v}(t)\rangle + \frac{1}{2}\langle L\mathbf{u}(t), \mathbf{u}(t)\rangle$is the total energy of the elastic body;
- $\langle \mathbf{f}(t), \mathbf{v}(t)\rangle$ is the rate at which work is done by the external forces;
- $\langle M\mathbf{v}(t), \mathbf{v}(t)\rangle$ is the rate of energy loss due to viscosity;
- $\langle \gamma_T^* \xi(t), \mathbf{v}(t)\rangle$ is the rate of energy loss due to friction.

It remains to show that frictional energy loss obeys the maximum dissipation principle.

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Maximum Dissipation

Claim: $\langle \gamma_T^* \xi(t), \mathbf{v}(t) \rangle = \int_{\Gamma_C} \mu(|\mathbf{v}_T|)|(\mathscr{R}\sigma)_n|(|\mathbf{v}_T|)d\Gamma$ for a.a. t.
That is, frictional forces are strictly dissipative and the principle of maximum dissipation is satisfied.

Proof:

Step 1: Kuttler and Shillor showed that for all $\mathbf{w} \in \mathscr{K}_{\mathbf{u}}$,

$$\int_0^T \langle \gamma_T^* \xi, \mathbf{w} \rangle \, dt \leq \int_0^T \int_{\Gamma_C} \mu(|\mathbf{v}_T|)|(\mathscr{R}\sigma)_n|(|\mathbf{v}_T + \mathbf{w}_T| - |\mathbf{v}_T|) d\Gamma dt.$$

We can easily show

$$\int_a^b \langle \gamma_T^* \xi, \mathbf{w} \rangle \, dt \leq \int_a^b \int_{\Gamma_C} \mu(|\mathbf{v}_T|)|(\mathscr{R}\sigma)_n|(|\mathbf{v}_T + \mathbf{w}_T| - |\mathbf{v}_T|) d\Gamma dt$$

for any $a, b \in [0, T]$.

Maximum Dissipation

Step 2: Choose $\mathbf{w} = \mathbf{v}$ to get

$$\int_a^b \langle \gamma_T^* \xi, \mathbf{v} \rangle \, dt \leq \int_a^b \int_{\Gamma_C} \mu(|\mathbf{v}_T|) |(\mathscr{R}\sigma)_n| (|\mathbf{v}_T|) d\Gamma dt.$$

Step 3: Choose $\mathbf{w} = -\mathbf{v}$ to get

$$\int_a^b \langle \gamma_T^* \xi, \mathbf{v} \rangle \, dt \geq \int_a^b \int_{\Gamma_C} \mu(|\mathbf{v}_T|) |(\mathscr{R}\sigma)_n| (|\mathbf{v}_T|) d\Gamma dt$$

Then

$$\int_a^b \langle \gamma_T^* \xi, \mathbf{v} \rangle \, dt = \int_a^b \int_{\Gamma_C} \mu(|\mathbf{v}_T|) |(\mathscr{R}\sigma)_n| (|\mathbf{v}_T|) d\Gamma dt$$

for any $a, b \in [0, T]$. Hence,

$$\langle \gamma_T^* \xi, \mathbf{v} \rangle = \int_{\Gamma_C} \mu(|\mathbf{v}_T|) |(\mathscr{R}\sigma)_n| (|\mathbf{v}_T|) d\Gamma \qquad \text{for a.a. } t \in [0, T].$$

- Kuttler, K. and Shillor, M. *Dynamic contact with Signorini's condition and slip rate dependent friction.* Electronic Journal of Differential Equations, Vol. 2004(2004), No. 83.
- Stewart, D.E. *Energy balance for viscoelastic bodies in frictionless contact.* submitted to *Quarterly of Applied Mathematics*, February 2007.