An Intuitive Approach to Collision Modeling Using Complementarity

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Outline

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Why Model Collisions?

Accurate simulation of collisions are needed in many areas.

- Video Games
- Physically Based Simulators
- Robotics
- Movie Special Effects

A Simple Motion Problem

The position of a projectile t seconds after being fired from ground level at some initial speed v_0 under the influence of gravity can be described with the function

$$x(t) = -gt^2 + v_0 t.$$

This function is only valid until a collision occurs.

A Generalization of the Rigid Body Impact Problem

Given an arbitrary collection of rigid bodies in motion, we would like to simulate their movement so that

- the bodies obey Newton's Laws of motion;
- the bodies respect each other's boundaries;
- small changes in the input parameters do not significantly change the result.

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Newton's Laws of Motion

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- Objects in motion tend to stay in motion; objects at rest tend to stay at rest (unless acted upon by an exterior force).
- $F = ma$
- For every action, there is an equal and opposite reaction.

The Catch

When two bodies collide, they exert forces upon each other according to Newton's Third Law. This can cause nearly instantaneous changes in the velocities and accelerations of the objects. (i.e. we may no longer have a continuous system!)

A Modeling Constraint

Computers are limited to only consider a finite number of calculations. In order to simulate a continuous system, we must therefore discretize our system in both time and space.

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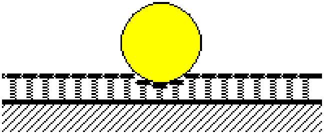
Penalty Methods

Penalty methods are widely used as a means to apply continuous mathematical methods to this inherently discontinuous problem. Instead of treating boundaries as rigid obstacles, we view them as a collection of stiff springs.



Inadequacies of Penalty Systems

The force exerted by an obstacle on a moving body is proportional to the extent of interpenetration. If the time-steps of the simulation are not appropriately small, a fast-moving object may penetrate deeply into the obstacle. The resulting spring force must be correspondingly large.

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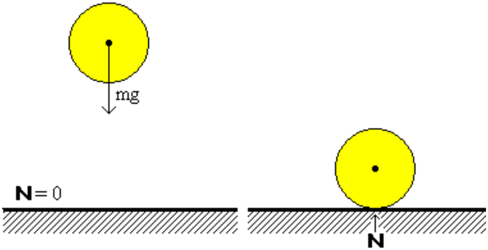
Simple Complementarity

When objects are not in contact, they exert no (impact) forces upon one another. That is, either the distance between them is zero (in contact) or the forces exerted are zero (no contact).



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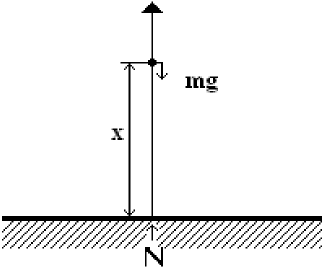
The One Dimensional Particle System

For a single particle in one dimension, we can formalize our intuition as

$$0 \leq x \perp N \geq 0,$$

where x is the distance to a fixed plane and N is the normal contact force exerted by the plane on the particle during impact.



Discretization and Numerical Approximation

To perform our simulations, we use an implicit midpoint approximation and time-stepping method. At time step l, our complementarity system is

$$0 \leq x^{l+1} \perp N^{l+1} \geq 0.$$

The goal will be to rewrite the left-side constraint in terms of values from the previous time step and the unknown force N^{l+1}.

Discretization and Numerical Approximation

We use the following approximations:

$$x^{l+1} \approx x^l + h\left(\gamma \dot{x}^{l+1} + (1-\gamma)\dot{x}^l\right)$$

$$\dot{x}^{l+1} \approx \dot{x}^l + h \cdot \frac{N^{l+1}}{m}$$

The parameter γ will be specified later.

Discretization and Numerical Approximation

We may now rewrite the complementarity system as

$$0 \leq x^l + h\dot{x}^l + \frac{h^2\gamma}{m}N^{l+1} \perp N^{l+1} \geq 0$$

Since N^{l+1} is our only unknown, our solution will be based on the values of x^l and $\dot{x}^l$, so that

$$N^{l+1} = \begin{cases} 0 & \text{if } x^l + h\dot{x}^l \geq 0 \\ -\frac{m}{h^2\gamma}\left(x^l + h\dot{x}^l\right) & \text{if } x^l + h\dot{x}^l < 0 \end{cases}$$

A Simple Example

Consider a particle of unit mass with center at an elevation of 1 and initial velocity of 1 m/s downward. Let's use a time-step of 0.1 seconds and choose $\gamma = 0.5$.

Step	Position	Velocity	$x^l + h\dot{x}^l - h^2 g$	N^{l+1}
1	1.000	-1.00	0.851	0
2	0.851	-1.98	0.604	0
3	0.604	-2.96	0.259	0
4	0.259	-3.94	**-0.184**	36.8
5	0	-1.24	**-0.173**	34.6
6	0	1.24	0.075	0
7	0.075	0.26	0.052	0
8	0.052	-0.72	**-0.069**	13.8
9	0	-0.32	**-0.081**	16.2

The Coefficient of Restitution

The parameter γ as used in previous slides is closely related to the coefficient of restitution, which describes a ratio of pre-impact and post-impact velocity.

- We choose $\gamma = \frac{1}{1+e}$, where e is the desired coefficient of restitution.
- For perfectly elastic impact, $e = 1$.
- For perfectly inelastic impact, $e = 0$.

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Complementarity in Three Dimensions

The one-dimensional model can be extended to three dimensions by treating the complementarity condition componentwise.

$$\begin{aligned} 0 \leq x_1 \quad &\perp \quad N_{x_1} \geq 0 \\ 0 \leq x_2 \quad &\perp \quad N_{x_2} \geq 0 \\ 0 \leq x_3 \quad &\perp \quad N_{x_3} \geq 0 \end{aligned}$$

In vector notation, we can write

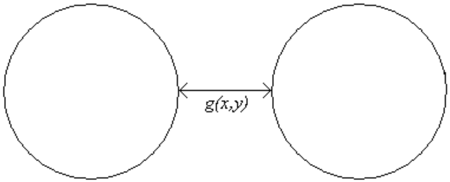
$$\mathbf{0} \leq \mathbf{x} \perp \mathbf{N} \geq \mathbf{0}.$$

Complementarity Between Arbitrary Objects

Further extension allows us to model impacts between two or more moving objects

$$\mathbf{0} \leq \mathbf{g}(\mathbf{x}, \mathbf{y}) \perp \mathbf{N}(\mathbf{x}, \mathbf{y}) \geq \mathbf{0},$$

where $\mathbf{g}(\mathbf{x}, \mathbf{y})$ is a “gap” function describing a distance between objects x and y and $\mathbf{N}$ is the force exerted on object x by object y.



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Contact Detection

In order for our complementarity scheme to work, we need an efficient method for determining when two objects are in contact.

- For simplicity, assume all objects are spheres or planes.
- For more complex objects, we can build “bounding” hierarchies.

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Sphere-Plane Contact

Given a sphere with center (x_0, y_0, z_0) and radius r, and a plane with equation $Ax + By + Cz + D = 0$, we can find the distance between the center and the plane to be

$$d = \frac{Ax_0 + By_0 + Cz_0 + D}{\sqrt{A^2 + B^2 + C^2}}.$$

If this distance is less than the radius of the sphere, the objects are in contact.

Sphere-Sphere Contact

Given two spheres with centers (x_0, y_0, z_0) and (x_1, y_1, z_1) and radii r_0 and r_1, respectively, we calculate the distance between the centers using simple Euclidean distance:

$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$$

If this distance is less than the sum $r_0 + r_1$, the spheres are in contact.

Predicting Impact

Given any two objects in our model, we determine whether the two objects collide *during the next time step*. That is, we tentatively update the position of each object, then calculate the gap between them.

- If there is no contact, leave the objects in their new positions.
- If there is contact, flag the pair of objects (we'll deal with them later) and search for other impacts.

Collision Resolution

Once a collision has been detected, we need to solve our complementarity system to determine how much force must be applied to avoid interpenetration of objects. Once this force is determined, we update the positions and velocities of each object using the update formulas:

$$x^{l+1} = x^l + h\left(\gamma \dot{x}^{l+1} + (1-\gamma)\dot{x}^l\right)$$

$$\dot{x}^{l+1} = \dot{x}^l + h \cdot \frac{N^{l+1}}{m}$$

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Complementarity Matrix Formulation

The actual complementarity formulation used in our simulator is velocity based rather than position based.

$$\mathbf{0} \leq \dot{\mathbf{x}}^{l+1} + hM^{-1}\mathbf{N}^{l+1} - h\mathbf{g} \perp \mathbf{N}^{l+1} \geq \mathbf{0}$$

We transform this into a matrix formulation of the form

$$\mathbf{0} \leq A\mathbf{n} + \mathbf{b} \perp \mathbf{n} \geq \mathbf{0}$$

Complementarity Solvers

Complementarity systems can be solved efficiently using several different methods. Some of the most popular methods are:

- Lemke's Algorithm–similar to the Simplex method for linear programs
  - solves complementarity systems exactly
  - worst-case exponential complexity
- Iterative Methods–like Gauss-Seidel or Jacobi Method
  - finds approximate solutions
  - parallelizable and generally quicker than Lemke's Algorithm

Simulation Examples

Let's look at a few different configurations of the model.

Possible Extensions

- Allow for rotations (simple generalization)
- Allow for more complex shapes (difficulty depends on shape)
- Add friction (hard)
- Allow for deformation of bodies (visco-elasticity)

Brogliato, B.
Nonsmooth Mechanics.
Springer, 1999.

Duvaut, G. and Lions, J.
Inequalities in Mechanics and Physics.
Springer, 1972.

Stronge, W.
Impact Mechanics.
Cambridge University Press, 2000.

Erleben, K.
Stable, Robust, and Versatile Multibody Dynamics Animation.
Doctoral Thesis, 2004.

Resources

Presentation references and materials can be found at the following websites:

http://www.math.uiowa.edu/~twendt/

http://www.math.uiowa.edu/~bgalluzz/