Overview. My research interests lie primarily in the study of modeling the dynamic contact between physical bodies, especially in the case of impact. One of the tools I've used to model contact conditions is the complementarity problem. In their simplest form, complementarity problems involve (for a given function $f\colon \mathbb{R}^n \to \mathbb{R}^n$) finding a vector z that is componentwise non-negative ("$z \geq 0$") such that $f(z)$ is also componentwise non-negative and $z^T f(z) = 0$. This can be written as

$$0 \leq z \quad \perp \quad f(z) \geq 0.$$

For example, in modeling the contact of a body with a rigid obstacle, both the separation and the normal contact force should be non-negative, but if one is positive, the other must be zero. That is,

$$0 \leq \text{separation} \quad \perp \quad \text{normal force} \geq 0.$$

These are essentially the conditions first proposed by Signorini in the 1930's for static contact problems. The first work in this field was done in 1860 by E.J. Routh, who modeled the impact of a metal rod with a fixed rigid obstacle using the one-dimensional wave equation with complementarity conditions for the end of the rod that can make contact. In 1933, Signorini formulated a model for static equilibrium of an elastic body in unilateral contact with a fixed rigid obstacle. Signorini's problem was first solved by Fichera [3] using variational inequalities. It remains a common practice to formulate dynamic complementarity problems as variational inequalities in order to prove existence of solutions.

My research uses complementarity problems and variational inequalities to examine contact and impact for various models. In general, my research can be summarized as follows:

(1) **Hybrid position-velocity based complementarity:**
In the study of impact dynamics, the complementarity formulation has been used in two ways. Traditional approaches have formulated complementarity in terms of relative position of a moving body to a fixed obstacle. Some recent formulation have been in terms of velocity of the moving body rather than its position. I am currently investigating a hybrid formulation using both position and velocity information in the complementarity formulation.

(2) **Energy balance in viscoelastic contact models with friction** :
I have established energy balance (i.e. accounting for all energy lost in a system) for the problem of a viscoelastic body in dynamic contact with a rigid foundation with a slip-rate dependent Coulomb friction model. These are the first energy results for Signorini's problem with friction. The methods can be extended to similar impact formulations.

(3) **Instabilities in contact models due to friction**:
For the problem of a two-dimensional elastic body in sliding contact with a rigid foundation, Renardy [9] showed that instabilities occur at all size scales. This problem has a similar formulation in three-dimensions, and I believe that we can show similar results.

(4) **Fractional index convolution complementarity problems**:
Closely related to classical complementarity problems, convolution complementarity problems often arise in the study of impact of elastic bodies. The results in joint work with Stewart [13] establish existence and uniqueness of solutions to a particular class of convolution complementarity problems.

1. Hybrid position-velocity based complementarity

It has been difficult for researchers to establish results for dynamic problems with Signorini's contact condition. Existence of solutions has been shown for viscoelastic bodies, but not for purely elastic bodies.

In the study of impact dynamics, the complementarity formulation has been used in two ways. Traditionally, complementarity has been enforced based on the relative position of the body to the obstacle. These so-called 'position based' complementarity formulations require that no interpenetration occur between the body and the obstacle. Position based complementarity models have been examined in [1], [4], [6], [7], [8], and others. A technical difficulty of using position based complementarity for the problem of viscoelastic contact with Coulomb friction is that it requires the definition of the trace of the stress on the contact surface, which is not generally well-defined. To avoid this problem, Cocou [1] and Kuttler and Shillor [5] use a regularization operator so that any irregularities in the boundary stress are smoothed out. With regularization, they are able to show existence, but not uniqueness, of solutions.

As an alternative to the traditional position based complementarity formulation, some researchers have experimented with a velocity based formulation. Here, the complementarity is of the form

$$0 \leq \text{velocity} \quad \perp \quad \text{normal force} \geq 0,$$

and is only 'turned on' when impact is imminent. This essentially requires that the velocity change directions immediately following an impact. Velocity based complementarity formulations have been used by Eck and Jarušek [2], among others. One advantage of these formulations is that they allow the system to be treated as parabolic rather than hyperbolic. Additionally, because the complementarity is written in terms of normal velocity rather than normal displacement, it is not necessary to regularize the surface stresses. Instead, we need a bound on the friction coefficient in order to show that solutions exist. As in the case of position based models, uniqueness of solutions is unknown.

More recently, I've examined the problem of an elastic or viscoelastic body in contact with a fixed rigid foundation using a hybrid complementarity formulation incorporating both position and velocity. The goal in this research is to avoid the need for regularization *and* eliminate the dependence upon a coefficient of friction bound. I am currently examining both the frictional and frictionless cases, and believe that I can show existence of solutions to both problems.

2. Energy Balance in viscoelastic contact models with friction

In addition to the difficulties in establishing existence and uniqueness for dynamic Signorini problems, there are only minimal results pertaining to energy conservation or energy balance (i.e. accounting for all energy lost in a system), and very few results for Signorini contact conditions involving Coulomb friction.

In [14], I was able to show that solutions to the problem of Kuttler and Shillor [5] obey the principle of maximum frictional dissipation and have energy balance. In particular, this energy balance shows that the impact itself is not responsible for any gain or loss of energy in the system. The proof of my result involved applying a lemma of Stewart [12] directly to solutions of Kuttler and Shillor's variational inequality formulation. This technique, referred to by Stewart as 'differentiating variational inequalities,' allows us to identify the terms representing kinetic and elastic potential energy, as well as viscous and frictional losses. This is the first proof of an energy balance for viscoelastic bodies in impact with any kind of friction model. Using techniques similar to the ones I used in [14], I believe that we can also show that solutions to the problem of Eck and Jarušek (velocity based complementarity) and solutions to my hybrid complementarity formulation have energy balance.

3. Instabilities due to friction

I am also interested in examining (or establishing) a correlation between the Lamé parameters of a material and the effective coefficient of sliding friction. In [9], Renardy was able to show the ill-posedness of a problem of frictional sliding of an elastic body against a rigid foundation in two dimensions. I have reason to believe that these results can be extended to more general scenarios, and that the occurrence of the instabilities (directly related to the ratio of the Lamé parameters) is a primary factor in the observed coefficient of sliding friction.

4. Fractional index convolution complementarity problems

Convolution complementarity problems have the form: given a kernel function k and a function q, find a function u such that

$$0 \leq u(t) \perp (k * u)(t) + q(t) \geq 0,$$

where orthogonality is interpreted as

$$\int_0^T u(t)^T \left[(k * u)(t) + q(t)\right] \, dt = 0.$$

A fractional index problem of this kind has $k(t) \sim K_0 t^{\alpha - 1}$ for t small, with $0 < \alpha < 1$. These types of problems arise from problems arise from 'switching' systems or discontinuous ODEs. Building off of previous work by Stewart [10, 11], I was able to show uniqueness of solutions in my portion of a joint work with Stewart [13].

Future Research. I intend to continue studying impact and collisions of physical bodies. In particular, I am interested in examining the possibility of energy balance for many of the existing impact models. As mentioned in section 2, energy results for impact problems have been elusive. I believe that I can apply the method developed in [14] to other impact models, thereby strengthening physical plausibility of those models.

Additionally, I am interested in the uniqueness (or non-uniqueness) of solutions in problems of elastic and viscoelastic body impact problems in both the frictional and frictionless cases. Current research gives little indication of whether we can expect uniqueness from the Signorini style contact model. I hope to examine different types of complementarity formulations in order to gain insight into what conditions are necessary for uniqueness, or perhaps which conditions can be shown to yield non-unique solutions.

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