FRACTIONAL INDEX CONVOLUTION COMPLEMENTARITY PROBLEMS

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ABSTRACT. Convolution complementarity problems have the form: given a kernel function k and a function q, find a function u such that $u(t) \geq 0$, $(k * u)(t) + q(t) \geq 0$ for (almost) all t, and where $\int_0^T u(t)^T \left[(k * u)(t) + q(t)\right]\, dt = 0$. A fractional index problem of this kind has $k(t) \sim K_0\, t^{\alpha-1}$ for t small, with $0 < \alpha < 1$. Such problems are shown to have unique solutions under mild conditions. Numerical results are presented for a simple example.

1. INTRODUCTION

Consider the heat equation in a rod with a heat source at $x = 0$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(t)\, \delta(x), \tag{1.1}$$

$$u(0, x) = u_0(x). \tag{1.2}$$

Suppose also that the heat source has unbounded heat output, but that it is connected to a thermostat that turns it off when the temperature exceeds a threshold u^*. This can be represented by the following complementarity condition:

$$0 \leq u(t, 0) \quad \perp \quad f(t) \geq 0 \qquad \text{for almost all } t. \tag{1.3}$$

Here we use "$u(t, 0) \perp f(t)$" to mean that $u(t, 0) \cdot f(t) = 0$. Using Green's function for the heat equation $G(t, x) = (2\pi t)^{-1/2} \exp(-x^2/2t)$ we can reduce this to a problem in time only, since

$$u(t, x) = \int_{-\infty}^{+\infty} G(t, x - z)\, u_0(z)\, dz + \int_0^t G(t - \tau, x)\, f(\tau)\, d\tau.$$

This means that

$$u(t, 0) = \int_{-\infty}^{+\infty} G(t, -z)\, u_0(z)\, dz + \int_0^t G(t - \tau, 0)\, f(\tau)\, d\tau.$$

Putting $\varphi(t) = \int_{-\infty}^{+\infty} G(t, -z)\, u_0(z)\, dz$ and $k(t) = G(t, 0) = (2\pi t)^{-1/2}$, (1.3) becomes

$$0 \leq \varphi(t) + (k * f)(t) \quad \perp \quad f(t) \geq 0, \tag{1.4}$$

which is a Convolution Complementarity Problem (CCP) in f. CCP's were first used by Petrov and Schatzman in [4], although the idea was studied more systematically in [7].

In this paper we will consider CCP's which have the form: find $u\colon [0, T] \to \mathbb{R}^n$ satisfying

$$0 \leq u(t) \quad \perp \quad (k * u)(t) + q(t) \geq 0 \qquad \text{for (almost) all } t \in [0, T], \tag{1.5}$$

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where $k\colon [0,T] \to \mathbb{R}^{n\times n}$ and $q\colon [0,T] \to \mathbb{R}^n$ and

$$(k * u)(t) = \int_0^t k(t-\tau)\, u(\tau)\, d\tau$$

is the convolution of k and u. Note that we assume that we extend the domain of k to all of $\mathbb{R}$ by setting $k(t) = 0$ for $t < 0$. Of particular importance in the study of these problems is the *index* of a CCP. This is (loosely) defined to be the minimum $m \geq 0$ where $d^m k/dt^m(t)$ has a Dirac-δ function at $t = 0$. Thus if $k(t) = \delta(t)$ we have a problem with index zero (the CCP becomes $0 \leq u(t) \perp u(t) + q(t) \geq 0$); if $k(t) = H(t)$, the Heaviside function $H(t) = 1$ if $t > 0$ and $H(t) = 0$ if $t < 0$, then we have a problem with index one; if $k(t) = t_+ := \max(t, 0)$ then we have a problem with index two. For the "thermostat" problem described in (1.1, 1.2, 1.3) the index of the CCP we obtained (1.4) is $1/2$. This index is closely related to the index of Linear Complementarity Systems (LCS's) [2] and Differential Algebraic Equations (DAE's) [1]. Careful comparison will reveal that the DAE index is one more than the corresponding CCP index as described here.

In this paper we will consider fractional index CCP's, which we will take to have kernel functions of the form

$$k(t) = \psi_\alpha(t)\, K_0 + k_1(t), \qquad 0 < \alpha < 1, \tag{1.6}$$

where

$$\psi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \qquad \text{for all } \alpha > 0, \tag{1.7}$$

K_0 is symmetric positive definite, and k_1 suitably "nice" in a sense yet to be precisely defined.

Note that the Laplace transform of ψ_α is

$$\mathcal{L}\psi_\alpha(s) = s^{-\alpha}.$$

By using the Laplace transform as a definition rather than a theorem, we can extend ψ_α to allow any real value of α by allowing ψ_α to be a distribution; for $\alpha > 0$, if $f \in C^\infty$, $\psi_{-\alpha} * f = \psi_{\lceil\alpha\rceil-\alpha} * f^{(\lceil\alpha\rceil)}$ with $f^{(k)}$ being the kth derivative of f. This means that $\psi_\alpha * \psi_\beta = \psi_{\alpha+\beta}$ in the sense of distributions for all α, $\beta \in \mathbb{R}$. Since $\psi_1 * f(t) = \int_0^t f(\tau)\, d\tau$, we see that $\psi_{-k} = \delta^{(k)}$ and $\psi_{-k} * f = f^{(k)}$ for any integer $k \geq 0$. If we identify "$f \mapsto \psi_{-\alpha} * f$" as the αth derivative operator for any real α, then the index of CCP (1.5) with kernel (1.6) is α. Thus choosing $\alpha \in (0,1)$ gives us a CCP with fractional index.

CCP's with index zero and index one have been studied in [7], while Petrov and Schatzman study an index one-and-a-half problem in [4]. Linearized mechanical impact problems for rigid bodies can be formulated as index two CCP's. Existence and uniqueness results are demonstrated in [7] for index zero and index one problems; Petrov and Schatzman prove existence but not uniqueness for their specific problem in [4]. Mechanical impact problems for rigid bodies have solutions, but not uniqueness as they require a coefficient of restitution [6], even in the frictionless case.

While in general smaller indices result in easier problems, the assumptions for the existence theorem for index-one CCP's in [7] include the requirement that k has bounded variation in some interval $[0,T]$ with $T > 0$. This assumption cannot be satisfied for $K_0\, \psi_\alpha(t)$ as ψ_α is unbounded in a neighborhood to the right of zero.

Higher-index convolution complementarity problems with the form $k(t) \sim K_0\,\psi_{1+\alpha}(t) = K_0\,t^{\alpha}/\Gamma(1+\alpha)$, $0 < \alpha < 1$ for $t \downarrow 0$ have been investigated by Petrov and Schatzman, at least for the case $\alpha = 1/2$ in the context of the impact of a visco-elastic rod on a rigid table. These can also be shown to have solutions under mild conditions using similar techniques in [8].

In this paper we establish existence and uniqueness for fractional index CCP's with index between zero and one under mild conditions.

1.1. **Preliminaries.** Throughout this paper we assume that $\alpha \in (0,1)$. We will use χ_E to denote the characteristic function of a set E ($\chi_E(t) = 1$ if $t \in E$ and $\chi_E(t) = 0$ otherwise), and $\mathcal{F}f(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} f(t)\,dt$ to denote the Fourier transform of a function f. We will also generalize the complementarity condition beyond the standard condition $0 \leq a \perp b \geq 0$ to the generalized condition $\mathcal{C} \ni a \perp b \in \mathcal{C}^*$ where $\mathcal{C}$ is a closed convex cone, and $\mathcal{C}^*$ its dual cone:

$$\mathcal{C}^* = \{\, w \mid w^T z \geq 0 \quad \text{for all } z \in \mathcal{C}\,\}.$$

Note that if $\mathcal{C} = \mathbb{R}^n_+$ then $\mathcal{C}^* = \mathbb{R}^n_+$. This is an example of a self-dual cone.

2. SHORT-TIME EXISTENCE OF SOLUTIONS

The approach we take to establishing the existence of solutions is two-fold: first we "regularize" the fractional index CCP by adding a multiple of the Dirac-δ function to the kernel so that solutions exist and are well-behaved; we then use a result from the differentiation of complementarity problems [8] to establish uniform bounds on the solutions of the approximating problem, which leads to convergence of a subsequence and the existence of a solution.

Theorem 2.1. *Let $k(t) = K_0\psi_\alpha(t) + k_1(t)$ for $t > 0$, with K_0 symmetric positive definite, k_1 of locally bounded variation, $q \in H^{+\alpha/2}(0,T)$, and $\mathcal{C}$ a closed convex cone. Then the CCP*

$$\mathcal{C} \ni u(t) \quad \perp \quad (k * u)(t) + q(t) \in \mathcal{C}^* \qquad \textit{for almost all } t$$

has a solution $u \in H^{-\alpha/2}(0,T)$.

Before proceeding with the proof, we propose two lemmas and a theorem which will be used to establish existence, and later uniqueness of solutions.

Lemma 2.2. *For any complex matrix G and complex vector z,*

$$Re\ \left[\overline{z}^T G z\right] = \overline{z}^T\,\frac{1}{2}\,\left(G + \overline{G}^T\right) z$$

Proof. By definition, we have

$$\begin{aligned} Re[\overline{z}^T G z] &= Re[\overline{\overline{z}^T G z}^T] \\ &= Re[\overline{z}^T \overline{G}^T z]. \end{aligned}$$

Since the matrix $G + \overline{G}^T$ is Hermitian, $Re[\overline{z}^T G z] = Re[\overline{z}^T \frac{1}{2}(G + \overline{G}^T) z] = \overline{z}^T \frac{1}{2}(G + \overline{G}^T) z$. □

In addition to this basic matrix fact, we will also need to bound the norm of the Fourier transform of a function of bounded variation with compact support. We will later use this result to bound part of our kernel function.

Lemma 2.3. *Let k be a function of bounded variation with supp $k \subset [0,T]$. Then*

$$\|\mathcal{F}k(\omega)\| \leq \min\left(T\,\|k\|_{L^\infty(0,T)}, \frac{C(T)}{|\omega|}\right)$$

where $C(T) := \bigvee_0^T k(t) + \|k(0^+)\| + \|k(T^-)\|$.

Proof. As an initial bound, we examine the Fourier transform of k,

$$\begin{aligned}
\|\mathcal{F}k(\omega)\| &= \left\|\int_0^T e^{-i\omega t}k(t)dt\right\| \\
&\leq \int_0^T \left\|e^{-i\omega t}k(t)\right\| dt \\
&= \int_0^T \|k(t)\|\, dt \\
&\leq T\,\|k\|_{L^\infty(0,T)} \qquad (2.1)
\end{aligned}$$

Since k has locally bounded variation, $\|k\|_{L^\infty(0,T)}$ is finite so that (2.1) is a meaningful bound.

For a second bound, we notice that since k has locally bounded variation, k' is a measure. Therefore,

$$\int_0^T \|k'\|\, dt = \bigvee_0^T k(t).$$

Again we consider the Fourier transform

$$\mathcal{F}k(\omega) = \int_0^T e^{-i\omega t}k(t)dt.$$

Integrating by parts yields

$$\mathcal{F}k(\omega) = \frac{-i}{\omega}\int_0^T e^{-i\omega t}k'(t)dt.$$

So,

$$\begin{aligned}
\|\mathcal{F}k(\omega)\| &= \left\|\frac{-i}{\omega}\int_0^T e^{-i\omega t}k'(t)dt\right\| \\
&\leq \frac{1}{|\omega|}\int_0^T \|e^{-i\omega t}k'(t)\|dt \\
&= \frac{1}{|\omega|}\int_{-\infty}^{+\infty} \|k'(t)\|dt \\
&= \frac{1}{|\omega|}\bigvee_{-\epsilon}^{T+\epsilon} k(t) \\
&= \frac{1}{|\omega|}\left(\bigvee_0^T k(t) + \|k(0^+)\| + \|k(T^-)\|\right) \\
&:= \frac{C(T)}{|\omega|}. \qquad (2.2)
\end{aligned}$$

Together, (2.1) and (2.2) give

$$\|\mathcal{F}k(\omega)\| \leq \min\left(T\|k\|, \frac{C(T)}{|\omega|}\right).$$

□

Theorem 2.4. *Let k be a real valued function of the form $k = K_0\psi_\alpha + k_1$ where K_0 is symmetric positive definite and k_1 is of locally bounded variation. Assume $f \in H^{-\alpha/2}(0,T)$ and supp $f \subset [0,T]$. Then*

$$\|f\|^2_{H^{-\alpha/2}(0,T)} \leq \frac{2}{\lambda_{\min}(K_0)\cos(\frac{\alpha\pi}{2})}\left\|\int_{-\infty}^{+\infty}\overline{\mathcal{F}f(\omega)}^T\mathcal{F}k(\omega)\mathcal{F}f(\omega)\,d\omega\right\|.$$

Proof. Since k is real valued, we first note that

$$\begin{aligned}\int_{-\infty}^{+\infty}\overline{\mathcal{F}f(\omega)}^T\mathcal{F}k(\omega)\mathcal{F}f(\omega)\,d\omega &= Re\int_{-\infty}^{+\infty}\overline{\mathcal{F}f(\omega)}^T\mathcal{F}k(\omega)\mathcal{F}f(\omega)\,d\omega \\ &= \int_{-\infty}^{+\infty} Re\,[\overline{\mathcal{F}f(\omega)}^T\mathcal{F}k(\omega)\mathcal{F}f(\omega)]\,d\omega. \qquad (2.3)\end{aligned}$$

To obtain an approximation of this integral, we now split $\mathcal{F}k(\omega)$ into the dominant and dominated part (from $K_0\psi_\alpha$ and $k_1$respectively).

$$\int_{-\infty}^{+\infty} Re\,\overline{\mathcal{F}f(\omega)}^T\mathcal{F}k(\omega)\mathcal{F}f(\omega)\,d\omega = \int_{-\infty}^{+\infty} Re\,\overline{\mathcal{F}f(\omega)}^T\left(\mathcal{F}K_0(\omega) + \mathcal{F}k_1(\omega)\right)\mathcal{F}f(\omega)\,d\omega$$

In the dominant term, $\mathcal{F}(K_0\psi_\alpha)(\omega) = K_0(i\omega)^{-\alpha}$ so that its Hermitian part is

$$\begin{aligned}\frac{1}{2}\left(K_0(i\omega)^{-\alpha} + \overline{K_0(i\omega)^{-\alpha}}^T\right) &= \frac{1}{2}K_0\left((i\omega)^{-\alpha} + (-i\omega)^{-\alpha}\right) \\ &= K_0\,Re\,(i\omega)^{-\alpha} = \cos(\frac{\alpha\pi}{2})K_0\,|\omega|^{-\alpha}.\end{aligned}$$

By applying Lemma 2.2, we see

$$\begin{aligned}\int_{-\infty}^{+\infty} Re\,\overline{\mathcal{F}f(\omega)}^T\mathcal{F}K_0(\omega)\mathcal{F}f(\omega)\,d\omega &= \int_{-\infty}^{+\infty}\overline{\mathcal{F}f(\omega)}^T\frac{1}{2}\left(\mathcal{F}(K_0\psi_\alpha)(\omega) + \overline{\mathcal{F}(K_0\psi_\alpha)(\omega)}^T\right)\mathcal{F}f(\omega)\,d\omega \\ &= \cos\left(\frac{\alpha\pi}{2}\right)\int_{-\infty}^{+\infty}|\omega|^{-\alpha}\,\overline{\mathcal{F}f(\omega)}^T K_0\mathcal{F}f(\omega)\,d\omega \\ &\geq \lambda_{\min}(K_0)\,\cos\left(\frac{\alpha\pi}{2}\right)\int_{-\infty}^{+\infty}|\omega|^{-\alpha}\,\|\mathcal{F}f(\omega)\|^2\,d\omega \qquad (2.4)\end{aligned}$$

since K_0 is symmetric positive definite.

Next we examine the dominated term. Since k_1 has locally bounded variation and supp $k_1 \subset [0,T]$, Lemma 2.3 gives us

$$\|\mathcal{F}k_1(\omega)\| \leq \min\left(T\|k_1\|_{L^\infty(0,T)}, \frac{C(T)}{|\omega|}\right).$$

Multiplying both sides by $|\omega|^\alpha$ yields

$$\|\mathcal{F}k_1(\omega)\||\omega|^\alpha \leq \min\left(T\|k_1\|_{L^\infty(0,T)}|\omega|^\alpha, C(T)|\omega|^{\alpha-1}\right).$$

For fixed T, we may find an explicit bound on this quantity:

$$\min\left(T\|k_1\|_{L^\infty(0,T)}|\omega|^\alpha, C(T)|\omega|^{1-\alpha}\right) \leq \left(C(T)^\alpha\left(T\|k_1\|\right)^{1-\alpha}\right)$$

Therefore, if we choose $T > 0$ small enough, we may make $\sup_\omega \|\mathcal{F}k_1(\omega)\|$ as small as we like. In particular, we choose $T > 0$ so that

$$\|\mathcal{F}k_1(\omega)\| \leq \frac{1}{2}\lambda_{\min}(K_0)\,\cos\left(\mp\frac{\alpha\pi}{2}\right)\,|\omega^{-\alpha}| \qquad \text{for all } \omega. \tag{2.5}$$

Using (2.4) together with (2.5) gives

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \overline{\mathcal{F}f(\omega)}^T \mathcal{F}k(\omega)\mathcal{F}f(\omega)\,d\omega \\
&= \int_{-\infty}^{+\infty} \overline{\mathcal{F}f(\omega)}^T \mathcal{F}K_0(\omega)\mathcal{F}f(\omega)\,d\omega + \int_{-\infty}^{+\infty} \overline{\mathcal{F}f(\omega)}^T \mathcal{F}k_1(\omega)\mathcal{F}f(\omega)\,d\omega \\
&\geq \lambda_{\min}(K_0)\,\cos\left(\frac{\alpha\pi}{2}\right)\int_{-\infty}^{+\infty} |\omega|^{-\alpha}\,\|\mathcal{F}f(\omega)\|^2\,d\omega + \int_{-\infty}^{+\infty} \overline{\mathcal{F}f(\omega)}^T \mathcal{F}k_1(\omega)\mathcal{F}f(\omega)\,d\omega \\
&\geq \lambda_{\min}(K_0)\,\cos\left(\frac{\alpha\pi}{2}\right)\int_{-\infty}^{+\infty} |\omega|^{-\alpha}\,\|\mathcal{F}f(\omega)\|^2\,d\omega - \left\|\int_{-\infty}^{+\infty} \overline{\mathcal{F}f(\omega)}^T \mathcal{F}k_1(\omega)\mathcal{F}f(\omega)\,d\omega\right\|
\end{aligned}$$

$$= \frac{1}{2}\lambda_{\min}(K_0)\,\cos\left(\frac{\alpha\pi}{2}\right)\,\|f\|^2_{H^{-\alpha/2}(0,T)}. \tag{2.6}$$

Rearranging, we get

$$\|f\|^2_{H^{-\alpha/2}(0,T)} \leq \frac{2}{\lambda_{\min}(K_0)\cos(\frac{\alpha\pi}{2})}\left\|\int_{-\infty}^{+\infty} \overline{\mathcal{F}f(\omega)}^T \mathcal{F}k(\omega)\mathcal{F}f(\omega)\,d\omega\right\|$$

as desired. □

Now we are able to begin with our proof of Theorem 2.1.

Proof. Let

$$k_\epsilon(t) = \epsilon\,\delta(t) + K_0\,\psi_\alpha(t) + k_1(t) = \epsilon\,\delta(t) + k(t). \tag{2.7}$$

Assume without loss of generality that $q(t) = 0$ for $t < 0$. This is possible since $q \in H^{+\alpha/2}(\mathbb{R})$ and the Heaviside function is in $H^{+\alpha/2}(\mathbb{R})$. Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a non-negative C^∞ function with support contained in $[-1, +1]$ and integral one, and set $\varphi_\epsilon(t) = (1/\epsilon)\varphi(\epsilon t)$. Define $q_\epsilon := \varphi_\epsilon * q(\cdot - \epsilon)$. Then $q_\epsilon \in C^\infty$ and $q_\epsilon(0) = 0$. Further, $q_\epsilon \to q$ strongly in $H^{+\alpha/2}(\mathbb{R})$ as $\epsilon \to 0$. Let $u_\epsilon$be the solution of the index zero CCP

$$\mathcal{C} \ni u_\epsilon(t) \quad \perp \quad (k_\epsilon * u_\epsilon)(t) + q_\epsilon(t) \in \mathcal{C}^*. \tag{2.8}$$

Now note that

$$(k_\epsilon * u_\epsilon)(t) = \epsilon\,u_\epsilon(t) + K_0\,(\psi_\alpha * u_\epsilon)(t) + (k_1 * u_\epsilon)(t).$$

Thus (2.8) becomes

$$\mathcal{C} \ni u_\epsilon(t) \quad \perp \quad \epsilon\,u_\epsilon(t) + K_0\,(\psi_\alpha * u_\epsilon)(t) + (k_1 * u_\epsilon)(t) + q_\epsilon(t) \in \mathcal{C}^* \tag{2.9}$$

for all t, and by taking the inner product of the two sides we can apply variants of the Gronwall lemma to show that u_ϵ must be uniformly bounded on $[0, T]$. Notice that q_ϵ is Lipschitz and that $q_\epsilon(0) = 0$. That solutions exist and are unique for (2.9) is shown in [7]. Lipschitz continuity of u_ϵ follows by the method of construction used in [7] for Lipschitz q_ϵ.

Since $u_\epsilon(t)^T[\epsilon u_\epsilon(t) + (k * u_\epsilon)(t) + q_\epsilon(t)] = 0$ for all t and $\epsilon > 0$, it follows that $u_\epsilon(t)^T[(k * u_\epsilon)(t) + q_\epsilon(t)] \leq 0$, and hence

$$u_\epsilon(t)^T(k * u_\epsilon)(t) \leq -u_\epsilon(t)^T q_\epsilon(t) \qquad \text{for all } t.$$

Integrating over $[0,T]$ gives

$$\int_0^T u_\epsilon(t)^T(k*u_\epsilon)(t) \leq -\int_0^T u_\epsilon(t)^T q_\epsilon(t).$$

If we let $f=\chi_{[0,T]}u_\epsilon$, then we may write

$$\int_{-\infty}^{+\infty} f(t)^T(k*f)(t)\,dt \leq -\int_{-\infty}^{+\infty} f(t)^T q_\epsilon(t)\,dt.$$

Taking Fourier transforms and using Parseval's theorem [3, Thm. 5.3, p. 243] we see that

$$\int_{-\infty}^{+\infty} \overline{\mathcal{F}f(\omega)}^T\mathcal{F}k(\omega)\mathcal{F}f(\omega)\,d\omega \leq -\int_{-\infty}^{+\infty} \overline{\mathcal{F}f(\omega)}^T\mathcal{F}q_\epsilon(\omega)\,d\omega.$$

Applying Theorem 2.4 then gives us

$$\begin{aligned}\|f\|^2_{H^{-\alpha/2}(0,T)} &\leq \frac{2}{\lambda_{min}(K_0)\cos(\frac{\alpha\pi}{2})}\left\|\int_{-\infty}^{+\infty}\overline{\mathcal{F}f(\omega)}^T\mathcal{F}k(\omega)\mathcal{F}f(\omega)\,d\omega\right\| \\ &\leq \frac{2}{\lambda_{min}(K_0)\cos(\frac{\alpha\pi}{2})}\left\|\int_{-\infty}^{+\infty}\overline{\mathcal{F}f(\omega)}^T\mathcal{F}q_\epsilon(\omega)\,d\omega\right\| \\ &\leq \frac{2}{\lambda_{min}(K_0)\cos(\frac{\alpha\pi}{2})}\|q_\epsilon\|_{H^{+\alpha/2}(0,T)}\|f\|_{H^{-\alpha/2}(0,T)}.\end{aligned}$$

Thus we find that $\|f\|_{H^{-\alpha/2}(0,T)}=\|u_\epsilon\|_{H^{-\alpha/2}(0,T)}$ is bounded independently of $\epsilon>0$ since $\|q_\epsilon\|_{H^{+\alpha/2}(0,T)}\to\|q\|_{H^{+\alpha/2}(0,T)}$. As $H^{-\alpha/2}(0,T)$ is a Hilbert space, and therefore also reflexive, by Alaoglu's theorem [3, p. 71] there is a weakly convergent subsequence (also denoted u_ϵ). Now, (2.6) makes the mapping $f\mapsto(1/2\pi)\int_{-\infty}^{+\infty}\overline{\mathcal{F}f(\omega)}^T\mathcal{F}k(\omega)\mathcal{F}f(\omega)d\omega = \int_0^T u_\epsilon^T(k*u_\epsilon)$ a continuous convex quadratic function on $H^{-\alpha/2}(0,T)$. Applying Mazur's lemma [3, p. 88] shows that this function is weakly lower semi-continuous.

Let $\widehat{u}$ be the weak limit of this subsequence. This implies that $k*u_\epsilon\to k*\widehat{u}$ weakly in $H^{+\alpha/2}(0,T)$. From lower semi-continuity, $\int_0^T\widehat{u}^T(k*\widehat{u})\leq\liminf_{\epsilon\to0}\int_0^T u_\epsilon^T(k*u_\epsilon)$. But

$$0 = \int_0^T u_\epsilon^T(k_\epsilon*u_\epsilon+q) = \epsilon\int_0^T\|u_\epsilon\|^2+\int_0^T u_\epsilon^T(k*u_\epsilon)+\int_0^T u_\epsilon^T q. \tag{2.10}$$

In the subsequence already identified, $\int_0^T u_\epsilon^T q_\epsilon\to\int_0^T\widehat{u}^T q$ as $u_\epsilon\rightharpoonup\widehat{u}$ (weakly) in $H^{-\alpha/2}$ and $q_\epsilon\to q$ in $H^{+\alpha/2}$. Taking the liminf of (2.10) as $\epsilon\to0$ in this subsequence we find that

$$0\geq\int_0^T\widehat{u}^T(k*\widehat{u}+q). \tag{2.11}$$

But as $u_\epsilon(t)\in\mathcal{C}$ and $k_\epsilon*u_\epsilon(t)+q(t)=\epsilon\,u_\epsilon(t)+k*u_\epsilon(t)+q(t)\in\mathcal{C}^*$ for all $\epsilon>0$ and (almost) all t, we see that after taking weak limits in the subsequence and using Mazur's lemma again, $\widehat{u}(t)\in\mathcal{C}$ and $k*\widehat{u}(t)+q(t)\in\mathcal{C}^*$ for (almost) all t as $\mathcal{C}$ and $\mathcal{C}^*$ are closed convex sets. The only way this can be consistent with (2.11) is if $\int_0^T\widehat{u}(k*\widehat{u}+q)=0$. Thus $\widehat{u}$ satisfies the convolution complementarity problem

$$\mathcal{C}\ni u(t)\quad\perp(k*u)(t)+q(t)\in\mathcal{C}^*\qquad\text{for (almost) all } t,$$

as we wanted. □

3. IMPROVED REGULARITY

In this section we show that for $q \in H^{1+\alpha/2}(\mathbb{R}, \mathbb{R}^n)$, the solution $u \in H^{1-\alpha/2}(0, T; \mathbb{R}^n)$. This really amounts to showing that the solutions u_ϵ of (2.8) are uniformly bounded in $H^{1-\alpha/2}(0, T)$.

Theorem 3.1. *Under the assumptions of Theorem 2.1, if* $q \in H^{1+\alpha/2}(\mathbb{R}, \mathbb{R}^n)$, *then the solution* $u \in H^{1-\alpha/2}(0, T; \mathbb{R}^n)$.

Proof. It is shown in [8] that if a and b are absolutely continuous with $a' \in L^p(0, T)$, $b' \in L^{p'}(0, T)$, $1/p + 1/p' = 1$ and satisfy

$$\mathcal{C} \ni a(t) \quad \perp \quad b(t) \in \mathcal{C}^* \qquad \text{for all } t, \tag{3.1}$$

then $\int_c^d a'(t)^T b'(t) \le 0$ for all $0 \le c \le d \le T$. Applying this to (2.9) gives the following inequality for almost all t:

$$\epsilon \|u_\epsilon'(t)\|^2 + u_\epsilon'(t)^T K_0 (\psi_\alpha * u_\epsilon')(t) + u_\epsilon'(t)^T (k_1 * u_\epsilon')(t) + u_\epsilon'(t)^T q'(t) \le 0. \tag{3.2}$$

Clearly $\epsilon \|u_\epsilon'(t)\|^2 \ge 0$ so

$$u_\epsilon'(t)^T K_0 (\psi_\alpha * u_\epsilon')(t) + u_\epsilon'(t)^T (k_1 * u_\epsilon')(t) + u_\epsilon'(t)^T q'(t) \le 0.$$

Re-arranging gives

$$u_\epsilon'(t)^T K_0 (\psi_\alpha * u_\epsilon')(t) \le -u_\epsilon'(t)^T (k_1 * u_\epsilon')(t) - u_\epsilon'(t)^T q'(t).$$

Integrating over $[0, T]$ gives

$$\int_0^T u_\epsilon'(t)^T K_0 (\psi_\alpha * u_\epsilon')(t)\, dt \le -\int_0^T \left(u_\epsilon'(t)^T (k_1 * u_\epsilon')(t) + u_\epsilon'(t)^T q'(t) \right) dt.$$

The integral on the left can be estimated using Fourier transforms. Note that for $t \in [0, T]$, $(k * u_\epsilon')(t) = (k * \chi_{[0,T]} u_\epsilon')(t)$. Furthermore,

$$\begin{aligned} \int_0^T u_\epsilon'(t)^T (k * u_\epsilon')(t)\, dt &= \int_{-\infty}^{+\infty} (\chi_{[0,T]} u_\epsilon')(t)^T (k * \chi_{[0,T]} u_\epsilon')(t)\, dt, \quad \text{and} \\ \int_0^T u_\epsilon'(t)^T q'(t)\, dt &= \int_{-\infty}^{+\infty} (\chi_{[0,T]} u_\epsilon')(t)^T q'(t)\, dt. \end{aligned}$$

In what follows, we consider $f = \chi_{[0,T]} u_\epsilon'$. Let

$$\mathcal{F} f(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} f(t)\, dt \tag{3.3}$$

denote the Fourier transform of f. First we note that

$$\begin{aligned} \mathcal{F}\psi_\alpha(\omega) &= (i\omega)^{-\alpha} = (e^{\pm i\pi/2} |\omega|)^{-\alpha} \\ &= e^{\pm i\alpha\pi/2} |\omega|^{-\alpha}, \end{aligned} \tag{3.4}$$

so that $\operatorname{Re} \mathcal{F}\psi_\alpha(\omega) = \cos(\alpha\pi/2)\, |\omega|^{-\alpha}$. Note that we want $\operatorname{Re} \mathcal{F}\psi_\alpha > 0$, so it is natural to restrict $0 < \alpha < 1$. By Parseval's theorem [3, Thm. 5.3, p. 243],

$$\begin{aligned} \langle f,\, K_0 (\psi_\alpha * f) \rangle &= \operatorname{Re} \frac{1}{2\pi} \langle \mathcal{F} f,\, K_0\, \mathcal{F}\psi_\alpha\, \mathcal{F} f \rangle \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \overline{\mathcal{F} f(\omega)}^T K_0 \mathcal{F} f(\omega) \operatorname{Re}(\mathcal{F}\psi_\alpha(\omega))\, d\omega \\ &\ge \frac{\lambda_{\min}(K_0)}{2\pi} \int_{-\infty}^{+\infty} \|\mathcal{F} f(\omega)\|_2^2 \cos(\alpha\pi/2)\, |\omega|^{-\alpha}\, d\omega. \end{aligned} \tag{3.5}$$

On the other hand

$$\begin{aligned}|\langle f,\, k_1 * f\rangle| &= \left|\frac{1}{2\pi}\int_{-\infty}^{+\infty}\|\mathcal{F}f(\omega)\|_2^2\,\mathcal{F}k_1(\omega)\,d\omega\right| \\ &\le \frac{1}{2\pi}\int_{-\infty}^{+\infty}\|\mathcal{F}f(\omega)\|_2^2\,\min(T\,\|k\|_{L^\infty}\,,\,\operatorname{Var}k/|\omega|)\,d\omega. \qquad (3.6)\end{aligned}$$

Provided we choose $T > 0$ sufficiently small, there is a positive constant C_1 where

$$\lambda_{\min}(K_0)\cos(\alpha\pi/2)\,|\omega|^{-\alpha} - \min(T\,\|k\|_{L^\infty}\,,\,\operatorname{Var}k/|\omega|) \ge C_1|\omega|^{-\alpha}. \qquad (3.7)$$

To find a suitable value for $T > 0$, we note that $T\,\|k\|_{L^\infty} = \operatorname{Var}k/|\omega|$ when $|\omega| = \omega^* := \operatorname{Var}k/(T\,\|k\|_{L^\infty})$. So it suffices for $\lambda_{\min}(K_0)\cos(\alpha\pi/2) - T\,\|k\|_{L^\infty}(\omega^*)^\alpha = C_1 > 0$; that is, we want $C_1 = \lambda_{\min}(K_0)\cos(\alpha\pi/2) - (T\,\|k\|_{L^\infty})^{1-\alpha}(\operatorname{Var}k)^\alpha > 0$, which is clearly possible for $0 < \alpha < 1$ and sufficiently small $T > 0$.

This means that

$$\begin{aligned}&\frac{C_1}{2\pi}\int_{-\infty}^{+\infty}|\omega|^{-\alpha}\,\left\|\mathcal{F}(\chi_{[0,T]}u_\epsilon')(\omega)\right\|_2^2 \\ &\quad\le \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\|\mathcal{F}(\chi_{[0,T]}u_\epsilon')(\omega)\right\|_2\,\|\mathcal{F}(q')(\omega)\|_2\,d\omega \\ &\quad= \frac{1}{2\pi}\int_{-\infty}^{+\infty}|\omega|^{-\alpha/2}\left\|\mathcal{F}(\chi_{[0,T]}u_\epsilon')(\omega)\right\|_2\,|\omega|^{+\alpha/2}\,\|\mathcal{F}(q')(\omega)\|_2\,d\omega.\end{aligned}$$

By Cauchy–Schwartz, noting that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}|\omega|^{-\gamma}\,\|f(\omega)\|_2^2\,d\omega = \|f\|_{H^{-\gamma/2}}^2\,,$$

we get

$$C_1\,\left\|\chi_{[0,T]}u_\epsilon'\right\|_{H^{-\alpha/2}}^2 \le \left\|\chi_{[0,T]}u_\epsilon'\right\|_{H^{-\alpha/2}}\|q'\|_{H^{+\alpha/2}}\,.$$

Dividing gives $\left\|\chi_{[0,T]}u_\epsilon'\right\|_{H^{-\alpha/2}} \le \|q'\|_{H^{+\alpha/2}}/C_1$.

This gives uniform bounds on $\|u_\epsilon'\|_{H^{-\alpha/2}(0,T)}$ independently of ϵ. By the standard Sobolev imbedding theorems [5, p. 215], there is a convergent subsequence u_ϵ' as $\epsilon \downarrow 0$ in $H^{-\gamma}(0,T)$ for any $\gamma > \alpha/2$. The limit can be identified as u' where u is the limit of the subsequence u_ϵ in the same subsequence (which would converge in $H^{1-\gamma}(0,T)$). Since $0 < \alpha < 1$ we can choose $\alpha/2 < \gamma < 1/2$. This would ensure that $1-\gamma > 1/2$, and so $H^{1-\gamma}(0,T) \subset C[0,T]$; thus $u_\epsilon \to u$ uniformly in the subsequence.

Thus $k_\epsilon * u_\epsilon = \epsilon\,u_\epsilon + k * u_\epsilon \to k * u$ uniformly as $\epsilon \downarrow 0$. Since $k_\epsilon * u_\epsilon + q \ge 0$, we see that $k * u + q \ge 0$. Similarly, since $u_\epsilon \ge 0$ for all $\epsilon > 0$, in the limit we also have $u \ge 0$ by uniform convergence. Finally,

$$\begin{aligned}0 &= \langle u_\epsilon,\, k_\epsilon * u_\epsilon + q\rangle \\ &\to \langle u,\, k * u + q\rangle\,.\end{aligned}$$

Thus the limit satisfies the conditions for solving the CCP, and a solution exists for the fractional index CCP.

The solution u constructed is continuous, and in fact, $u \in H^{1-\alpha/2}(0,T)$. □

4. UNIQUENESS AND CONTINUOUS DEPENDENCE OF SOLUTIONS FOR INDEX $0<\alpha<1$

Here we establish uniqueness of solutions to CCP's of fractional index $0<\alpha<1$ by applying Theorem 2.4 to show that any two solutions must be identical for almost all t.

Theorem 4.1. *Under the assumptions of Theorem 2.1, any solution u to the CCP*

$$\mathcal{C} \ni u(t) \quad \perp \quad (k*u)(t)+q(t) \in \mathcal{C}^* \qquad \textit{for almost all } t$$

is unique.

Proof. Suppose u_1, u_2 are two solutions of the CCP

$$0 \le u(t) \quad \perp \quad (k*u)(t)+q(t) \ge 0.$$

Also, suppose that

$$T^* = \sup\{\, t \mid u_1(\tau)=u_2(\tau) \text{ for all } \tau \le t\,\} < T.$$

We can shift the start of the CCP to T^* as follows: set $\widetilde{u}_1(t)=u_1(t+T^*)$ and $\widetilde{u}_2(t)=u_2(t+T^*)$. Then the CCP satisfied by $\widetilde{u}_1$ and $\widetilde{u}_2$ is

$$0 \le \widetilde{u}(t) \quad \perp \quad (k*\widetilde{u})(t)+\widetilde{q}(t) \ge 0$$

where $\widetilde{q}(t)=q(t+T^*)+\int_0^{T^*} k(t+T^*-\tau)\,u_1(\tau)\,d\tau$.

We can now restrict the problem to requiring only showing local uniqueness near $t=0$.

Set $y_i := (k*u_i)(t)+q(t)$ for i = 1, 2, and let $\omega(t)=u_2(t)-u_1(t)$, $\eta(t)=y_2(t)-y_1(t)$. Notice that

$$\begin{aligned}\omega(t)^T\eta(t) &= (u_2(t)-u_1(t))^T(y_2(t)-y_1(t))\\ &= u_2(t)^Ty_2(t)-u_2(t)^Ty_1(t)-u_1(t)^Ty_2(t)+u_1(t)^Ty_1(t)\\ &= -u_2(t)^Ty_1(t)-u_1(t)^Ty_2(t) \le 0.\end{aligned}$$

Therefore, we have

$$\begin{aligned}0 &\ge \int_0^T \omega(t)^T\eta(t)dt\\ &= \int_0^T \omega(t)^T\left((k*\omega)(t)\right)dt\\ &= \int_{-\infty}^{+\infty} \left(\chi_{[0,T]}\omega(t)\right)^T (k*\chi_{[0,T]}\omega)(t)dt.\end{aligned}$$

Let $f=\chi_{[0,T]}\omega$. Then

$$\begin{aligned}0 \ge \int_0^T \omega(t)^T\eta(t)dt &= \int_{-\infty}^{+\infty} f(t)^T\left((k*f)(t)\right)dt\\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{\mathcal{F}f(\omega)}^T\mathcal{F}k(\omega)\mathcal{F}f(\omega)d\omega.\end{aligned}$$

By Theorem 2.4, we must have

$$0 \quad \ge \quad \int_0^T \omega(t)^T\eta(t)dt \ge \frac{1}{4\pi}\cos(\frac{\alpha\pi}{2})\lambda_{\min}(K_0)\left\|f\right\|^2_{H^{-\alpha/2}(0,T)}$$

Since $\cos(\alpha\pi/2)\lambda_{\min}(K_0) > 0$, it follows that $\|f\|^2_{H^{-\alpha/2}(0,T)} \leq 0$, and thus $f(t) = 0$ for almost all t. Hence, $u_1(t) = u_2(t)$ for almost every $t \in [0, T]$, as desired. $\square$

REFERENCES

[1] K. E. BRENAN, S. L. CAMPBELL, AND L. R. PETZOLD, *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*, no. 14 in Classics in Applied Mathematics, SIAM Publ., Philadelphia, PA, 1996. Originally published by North Holland, 1989.

[2] W. P. M. H. HEEMELS, J. M. SCHUMACHER, AND S. WEILAND, *Linear complementarity systems*, SIAM J. Appl. Math, 60 (2000), pp. 1234–1269.

[3] S. LANG, *Real and Functional Analysis*, vol. 142 of Graduate Texts in Mathematics, Springer, Berlin, Heidelberg, New York, 2nd ed., 1993.

[4] A. PETROV AND M. SCHATZMAN, *Viscoélastodynamique monodimensionnelle avec conditions de Signorini*, Comptes Rendus Acad. Sci., Sér. I, 334 (2002), pp. 983–988.

[5] M. RENARDY AND R. C. ROGERS, *An Introduction to Partial Differential Equations*, vol. 13 of Texts in Applied Mathematics, Springer Verlag, New York, Berlin, Heidelberg, 1993.

[6] D. E. STEWART, *Rigid-body dynamics with friction and impact*, SIAM Review, 42 (2000), pp. 3–39.

[7] D. E. STEWART, *Convolution complementarity problems with application to impact problems*, IMA J. Applied Math., TBA (2005), p. TBA. Available through Advance Access; DOI 10.1093/imamat/hxh087.

[8] ———, *Differentiating complementarity problems and fractional index convolution complementarity problems*, Houston J. Mathematics, TBA (2006), pp. 1–21. To appear. Accepted October, 2005.