Rationalizing Fractions

An expression containing fractions is considered to be in simplified (or reduced) form if it is written as a single quotient with no radicals in the denominator. As a general rule of thumb, we'll consider any fraction expression with square roots in the denominator or with "fractions within fractions" to be unreduced. The process of simplifying an unreduced fraction is called *rationalizing* the fraction. To rationalize a given expression containing fractions, we'll generally want to perform the following steps:

1. Identify what's wrong with the given expression. Is there a square root in the denominator? Are there fractions embedded in other fractions? Are there terms that can still be combined or simplified?
2. If the expression has embedded fractions, we will try to write the numerator and denominator as single quotients by performing any addition or subtraction specified. (see example 1).
3. The expression should now be in the form of
$$\frac{(\text{fraction } 1)}{(\text{fraction } 2)}$$
We can perform this division operation by taking the product
$$(\text{fraction } 1) * \frac{1}{(\text{fraction } 2)}.$$
That is, multiply the numerator by the reciprocal of the denominator. The chosen fraction should now appear as a single quotient.
4. If there is a square root in the denominator, we must somehow remove it. For binomials, we will multiply by a term called the *conjugate*. For example, the conjugate of the term (a + b) is the term (a – b), and the conjugate of the term (a – b) is (a + b). By multiplying both the numerator and denominator by the conjugate of the denominator, the square root in the denominator will be removed.
5. After you have reduced your chosen piece, re-evaluate the original expression. If the expression is still unreduced, repeat steps 1-3 until it is reduced.

<u>Example 1:</u> Reduce $\frac{1+\frac{1}{x}}{2}$.

Step 1: This expression is not reduced, since it has a fraction embedded in the numerator.

Step 2: Choose to work with the numerator first.

$$1+\frac{1}{x}=\frac{x}{x}+\frac{1}{x}=\frac{x+1}{x}$$

Step 3:

$$\frac{1+\frac{1}{x}}{2}=\frac{\frac{x+1}{x}}{2}=\frac{x+1}{x}\cdot\frac{1}{2}=\frac{x+1}{2x}$$

The simplified form of the expression is $\frac{x+1}{2x}$.

Example 2: Reduce $\frac{1}{\sqrt{2}-5}$.

Step 1: This expression is not reduced since it has a square root in the denominator. We'll multiply the numerator and denominator by the conjugate of $\sqrt{2}-5$.

$$\frac{1}{\sqrt{2}-5}=\frac{1}{\sqrt{2}-5}\left(\frac{\sqrt{2}+5}{\sqrt{2}+5}\right)=\frac{\sqrt{2}+5}{-1}=-\sqrt{2}+5$$

Example 3: Reduce $x+\frac{x-1}{1-\sqrt{x}}$.

Step 1: This expression is not reduced since it has a square root in the denominator and two separate terms. We'll begin by fixing the square root term.

Step 2: Multiply the numerator and denominator of the second term by the conjugate of $1-\sqrt{x}$.

$$x+\frac{x-1}{1-\sqrt{x}}=x+\frac{x-1}{1-\sqrt{x}}\left(\frac{1+\sqrt{x}}{1+\sqrt{x}}\right)$$

$$=x+\frac{x+x\sqrt{x}-\sqrt{x}-1}{1-x}$$

Step 3: Now that the denominator of the second term has been rationalized, we can find a common denominator to add our terms.

$$x+\frac{x+x\sqrt{x}-\sqrt{x}-1}{1-x}=\frac{x}{1}\left(\frac{1-x}{1-x}\right)+\frac{x+x\sqrt{x}-\sqrt{x}-1}{1-x}$$

$$=\frac{x-x^2+x+x\sqrt{x}-\sqrt{x}-1}{1-x}$$

$$=\frac{-x^2+2x+x\sqrt{x}-\sqrt{x}-1}{1-x}$$

So, the expression is simplified as

$$x+\frac{x-1}{1-\sqrt{x}}=\frac{-x^2+2x+x\sqrt{x}-\sqrt{x}-1}{1-x}.$$