1. (5 points) Let $f(x) = \dfrac{x^2+3x+2}{x^2+x-2}$. Find all vertical and horizontal asymptotes of f.

$$f(x) = \frac{\cancel{(x+2)}(x+1)}{\cancel{(x+2)}(x-1)} \Rightarrow f \text{ has a vertical asymptote a } \underline{x=1}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2+3x+2}{x^2+x-2} = 1$$

$$\lim_{x\to-\infty} \frac{x^2+3x+2}{x^2+x-2} = 1$$

$\Rightarrow f$ has a Horizontal asymptote at $\underline{y=1}$

2. (10 points) Let $g(x) = x^3 - x$. Use the definition of the derivative to calculate $g'(x)$.

$$g'(x) = \lim_{h\to0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to0} \frac{(x+h)^3-(x+h)-(x^3-x)}{h}$$

$$= \lim_{h\to0} \frac{\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{x}-h-\cancel{x^3}+\cancel{x}}{h} = \lim_{h\to0} \frac{3x^2\cancel{h}+3x h^{\cancel{2}}+h^{\cancel{3}\,2}-\cancel{h}^{\,1}}{\cancel{h}}$$

$$= \lim_{h\to0} 3x^2+3xh+h^2-1 = 3x^2-1$$

3. For each of the following, calculate the derivative:

(a) (5 points) $f(x) = 4x^5 - 3x^4 + x^3 - 2x + 1$

$$f'(x) = 20x^4 - 12x^3 + 3x^2 - 2$$

(b) (8 points) $g(x) = \dfrac{3x-2}{\sqrt{2x+1}}$

$$g'(x) = \frac{\sqrt{2x+1}\,(3) - (3x-2)\frac{1}{2}(2x+1)^{-1/2}\cdot 2}{2x+1} = \frac{3\sqrt{2x+1} - (3x-2)(2x+1)^{-1/2}}{2x+1}$$

$$= \frac{3(2x+1)-(3x-2)}{(2x+1)^{3/2}} = \frac{6x+3-3x+2}{(\sqrt{2x+1})^3} = \frac{3x+5}{(\sqrt{2x+1})^3}$$

(c) (7 points) $h(x) = \sqrt{x \cdot \ln(x)}$

$$h'(x) = \frac{1}{2}(x\cdot \ln x)^{-1/2} \cdot \frac{d}{dx}(x \cdot \ln x) = \frac{1}{2}(x \ln x)^{-1/2} \cdot \left(x\,\frac{1}{x} + \ln x \cdot 1\right)$$

$$= \frac{1}{2}(x\ln x)^{-1/2} \cdot (1+\ln x) = \frac{1+\ln x}{2\sqrt{x\ln x}}$$

(d) (5 points) $i(x) = \sin(\tan x)$

$$i'(x) = \cos(\tan x) \cdot \frac{d}{dx}(\tan x)$$

$$= \cos(\tan x) \cdot \sec^2 x$$

4. (10 points) Use implicit differentiation to find y' if

$$x^2y - 3xy^2 + y = e^{2x}y$$

$$\frac{d}{dx}\left(x^2y - 3xy^2 + y\right) = \frac{d}{dx}\left(e^{2x}y\right)$$

$$x^2y' + 2xy - 6xyy' - 3y^2 + y' = e^{2x}y' + 2e^{2x}y$$

$$x^2y' - 6xyy' + y' - e^{2x}y' = 2e^{2x}y - 2xy + 3y^2$$

$$y'(x^2 - 6xy + 1 - e^{2x}) = 2e^{2x}y - 2xy + 3y^2$$

$$y' = \frac{2e^{2x}y - 2xy + 3y^2}{x^2 - 6xy + 1 - e^{2x}}$$

5. (10 points) Use logarithmic differentiation to find $f'(x)$ if

$$f(x) = \frac{(x^2+1)^4}{(2x^2+1)^3(3x-1)^5}.$$

$$y = \frac{(x^2+1)^4}{(2x^2+1)^3(3x-1)^5}$$

$$\ln y = \ln\left(\frac{(x^2+1)^4}{(2x^2+1)^3(3x-1)^5}\right)$$

$$\ln y = \ln(x^2+1)^4 - \ln(2x^2+1)^3 - \ln(3x-1)^5$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(4\ln(x^2+1) - 3\ln(2x^2+1) - 5\ln(3x-1)\right)$$

$$\frac{y'}{y} = \frac{4}{x^2+1}\cdot 2x - \frac{3}{2x^2+1}\cdot 4x - \frac{5}{3x-1}\cdot 3$$

$$y' = y\left(\frac{8x}{x^2+1} - \frac{12x}{2x^2+1} - \frac{15}{(3x-1)}\right)$$

$$y' = \frac{(x^2+1)^4}{(2x^2+1)^3(3x-1)^5}\left(\frac{8x}{x^2+1} - \frac{12x}{2x^2+1} - \frac{15}{3x-1}\right)$$

6. The half-life of Carbon 14 (^{14}C) is 5730 years.

(a) (5 points) Find a formula for the mass of ^{14}C that remains after t years given an initial mass of m_0.

$$m(t) = m_0e^{kt}$$

$$m(5730) = \tfrac{1}{2}m_0$$

$$m(5730) = m_0e^{5730k}$$

$$\tfrac{1}{2}\cancel{m_0} = \cancel{m_0}e^{5730k}$$

$$\tfrac{1}{2} = e^{5730k}$$

$$\ln \tfrac{1}{2} = 5730k$$

$$k = \frac{-\ln 2}{5730}$$

$$m(t) = m_0e^{(-\ln 2/5730)t}$$

(b) (5 points) A parchment is discovered that has 75% as much ^{14}C content as expected. Estimate the age of the parchment.

$$\frac{3}{4}\cancel{m_0} = \cancel{m_0}e^{-\ln 2/5730\, t}$$

$$\frac{3}{4} = e^{-\ln 2/5730\, t}$$

$$\ln(3/4) = -\frac{\ln 2}{5730}t$$

$$t = \ln\left(\frac{3}{4}\right)\cdot\frac{5730}{-\ln 2}$$

$$t = -\frac{\ln\left(\frac{3}{4}\right)\cdot 5730}{\ln 2}$$

7. The position of a particle is given by the function $s(t) = t^3 - 6t^2 + 9t$.

(a) (3 points) What is the velocity of the particle after 2 seconds?

$v(t) = s'(t) = 3t^2 - 12t + 9$

$v(2) = -3$

(b) (2 points) When is the particle moving forward?

when $v(t) > 0$

$v(t) = 3(t^2 - 4t + 3) = 3(t-1)(t-3)$

$t-1$: 0 – 1 + 3 +

$t-3$: – – +

\+ – +

Forward on $(0,1) \cup (3, \infty)$

(c) (5 points) When is the particle speeding up?

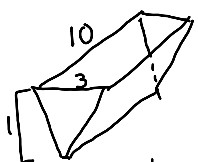
$a(t) = v'(t) = 6t - 12 = 6(t-2)$

$a(t) > 0$ for $t > 2$

$t-2$: 0 1 2 3; – – + +

speeding up on $(1,2) \cup (3, \infty)$

8. (10 points) A trough is 10 feet long and its ends have the shape of isosceles triangles that are 3 feet across at the top and have a height of 1 foot. If the trough is being filled with water at a rate of 12 cubic feet per minute, how fast is the water level rising when the water is 6 inches deep?



$V = \frac{1}{2} b \cdot h \cdot 10$

$V = \frac{1}{2}(3h)h \cdot 10$

$V = 15h^2$

$\frac{dV}{dt} = 12$

$\frac{b}{h} = \frac{3}{1}$

$b = 3h$

$\frac{dV}{dt} = 30h \frac{dh}{dt}$

$12 = 30\left(\frac{1}{2}\right)\frac{dh}{dt}$

$\frac{12}{15} = \frac{dh}{dt}$

$\frac{4}{5} = \frac{dh}{dt} \Rightarrow$ Water level is increasing at a rate of 0.8 ft/min

9. (10 points) Use differentials (or a linear approximation) to approximate the value of $\sqrt[3]{0.99}$.

Let $f(x) = \sqrt[3]{x}$

Choose $a = 1$, $x = 0.99$

$f'(x) = \frac{1}{3}x^{-2/3}$

$f'(a) = \frac{1}{3}$

$L(x) = f(a) + f'(a)(x-a)$

$L(0.99) = f(1) + f'(1)(0.99 - 1)$

$L(0.99) = 1 + \frac{1}{3}(-0.01)$

$= 1 - \frac{1}{300} = \frac{299}{300}$

$\sqrt[3]{0.99} \approx \frac{299}{300}$