1. (5 points) Verify the identity

$$\cosh^2 x - \sinh^2 x = 1$$

Recall $\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh^2 x - \sinh^2 x = [\frac{1}{2}(e^x + e^{-x})]^2 - [\frac{1}{2}(e^x - e^{-x})]^2$

$= \frac{1}{4}(e^{2x} + 2 + e^{-2x}) - \frac{1}{4}(e^{2x} - 2 + e^{-2x})$

$= \frac{1}{4}(2 - (-2)) = 1$ ✓

2. (5 points) State the Extreme Value Theorem.

If *f* is continuous on a closed interval, [*a*,*b*], then *f* attains an absolute maximum, *f*(*c*), and an absolute minimum, *f*(*d*) at some points *c*, *d*, within the interval [*a*,*b*].

3. (5 points) State the Mean Value Theorem.

Let *f* be a function such that:

1. *f* is continuous on the closed interval [*a*, *b*];
2. *f* is differentiable on the open interval (*a*, *b*)

Then there exists a point *c* on the interval (*a*, *b*) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

4. (10 points) Let $f(x) = 2x^3 - 3x^2 - 12x + 1$. Find the absolute maximum and minimum values of f on the interval $[-2, 3]$.

$f'(x) = 6x^2 - 6x - 12$

$0 = 6x^2 - 6x - 12$

$0 = x^2 - x - 2$

$0 = (x-2)(x+1)$

$x = 2$ or $x = -1$

$f(-2) = 2(-2)^3 - 3(-2)^2 - 12(-2) + 1$

$= -16 - 12 + 24 + 1 = -3$

$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 1$

$= -2 - 3 + 12 + 1 = 8$

$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 1$

$= 16 - 12 - 24 + 1 = -19$

$f(3) = 2(3)^3 - 3(3)^2 - 12(3) + 1$

$= 54 - 27 - 36 + 1 = -8$

Abs. Max. Value is 8

Abs. Min. Value is −19

5. (10 points) Suppose that $f'(x) = \sqrt{x}(6+5x)$ and $f(0) = 8$. Find f.

$f'(x) = 6x^{1/2} + 5x^{3/2}$

$f(x) = 6\left(\frac{2}{3}x^{3/2}\right) + 5\left(\frac{2}{5}x^{5/2}\right) + C$

$f(x) = 4x^{3/2} + 2x^{5/2} + C$

$f(0) = C = 8 \longrightarrow f(x) = 4x^{3/2} + 2x^{5/2} + 8$

6. (10 points) Let $g(x) = x^3 - x^2 - 6x + 2$. Verify that g satisfies the Mean Value Theorem on the interval $[-2, 2]$, and find all values c that satisfy the conclusion of the Mean Value Theorem.

1. g is continuous on $[-2,2]$ (g is polynomial.)
2. g in differentiable on $(-2,2)$ (g is polynomial)

So, there exists $c \in (-2,2)$ such that $g'(c) = \frac{g(b)-g(a)}{b-a}$

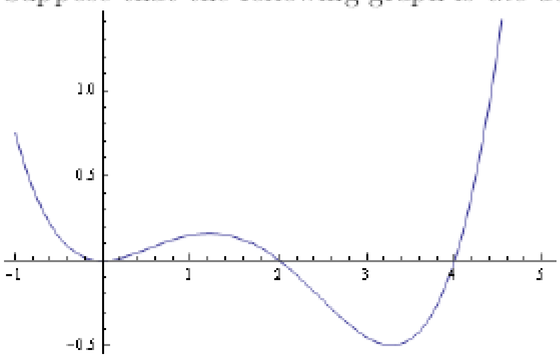
$\frac{g(2)-g(-2)}{2-(-2)} = \frac{[8-4-12+2]-[-8-4+12+2]}{4} = \frac{-6-2}{4} = -2$

Find $g'(c) = -2$.

$g'(c) = 3c^2 - 2c - 6 = -2 \Rightarrow 3c^2 - 2c - 4 = 0$

$c = \frac{2 \pm \sqrt{4+48}}{6} = \frac{2 \pm \sqrt{52}}{6}$

7. Suppose that the following graph is *the derivative* of a function $h(x)$.



(a) (4 points) On which interval(s) is h increasing?

h is increasing when $h' > 0. \Rightarrow (-1,2) \cup (4,5)$

(b) (4 points) On which interval(s)is h concave up?

h is concave up when h' is increasing $\Rightarrow (0, 1.2) \cup (3.3, 5)$

(c) (3 points) On which interval(s)is h'' decreasing?

h'' is decreasing when h' is concave down $\Rightarrow (0.5, 2.2)$

(d) (6 points) Describe the behavior of the graph of h near 0, 2, and 4? (be specific)

0, 2, 4 are critical numbers.

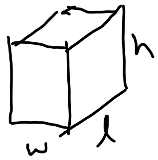
0 is an inflection point

2 is a local maximum

4 is a local minimum

8. A rectangular storage container with an open top is to have a volume of 10m^3. The length of the base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter.

(a) (5 points) Write the cost of producing the container as a function of the length of the base.



$V = 10 = w \cdot l \cdot h$

$l = 2w \Rightarrow w = \frac{l}{2}$

$10 = \frac{l^2}{2} h \Rightarrow h = \frac{20}{l^2}$

$\text{Cost} = 10wl + 6(2l \cdot h + 2wh)$

$C(l) = 5l^2 + 12l\left(\frac{20}{l^2}\right) + 12\left(\frac{l}{2}\right)\left(\frac{20}{l^2}\right)$

$C(l) = 5l^2 + \frac{240}{l} + \frac{120}{l} = 5l^2 + \frac{360}{l}$

(b) (5 points) Explain how you would use your function from part (a) to find the dimensions for the cheapest such container.

Take $C(l) = 5l^2 - \frac{360}{l}$. Calculate $C'(l)$, set equal to zero, and solve for critical numbers. Check to see which critical number gives the smallest value.

9. Let $f(x) = x^3 - 3x - 5$. $\rightarrow f'(x) = 3x^2 - 3$

(a) (5 points) Assuming $x_1 = 2$, calculate x_2 using Newton's Method.

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{-3}{9} = 2 + \frac{1}{3} = \frac{7}{3}$

(b) (5 points) Which initial values for Newton's Method must we avoid? Why?

Avoid where $f'(x) = 0$. Here, we want to avoid ± 1.

10. (8 points) Evaluate:

$$\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right).$$

$$\lim_{x \to 1} \left(\frac{x \ln x}{(x-1)\ln x} - \frac{x-1}{(x-1)\ln x} \right) = \lim_{x \to 1} \frac{x \ln x - x + 1}{x \ln x - \ln x}$$

by L'Hospital

$$= \lim_{x \to 1} \frac{1 + \ln x - 1}{1 + \ln x - \frac{1}{x}} = \lim_{x \to 1} \frac{\ln x}{1 + \ln x - \frac{1}{x}}$$

by L'Hospital

$$= \lim_{x \to 1} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{1}{2}$$