1. (10 points) Evaluate the following integral using the Riemann sum definition of integral:

$$\int_0^2 2x - 1\,dx$$

$$\int_0^2 2x-1\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} (2x_i - 1)\Delta x$$

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

$$x_i = a + i\Delta x = \frac{2i}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[2\left(\frac{2i}{n}\right) - 1\right]\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{8i}{n^2} - \frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{8}{n^2} \sum_{i=1}^{n} i - \frac{2}{n} \sum_{i=1}^{n} 1$$

$$= \lim_{n\to\infty} \frac{8}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{2}{\cancel{n}} \cdot \cancel{n}$$

$$= \lim_{n\to\infty} 4\left(\frac{n^2+n}{n^2}\right) - 2$$

$$= 4 - 2 = \boxed{2}$$

Aside

$$\int_0^2 2x-1\,dx$$

$$= x^2 - x \Big|_0^2 = (4-2) - 0$$

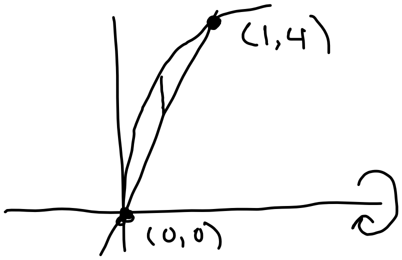
$$= 2$$ ✓

2. (10 points) State the Fundamental Theorem of Calculus (parts I and II).

3. (5 points) Let $f(x) = x^3 + x - 5$. Calculate the average value of f over the interval $[0, 2]$.

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2-0}\int_0^2 x^3 + x - 5\,dx$$
$$= \frac{1}{2}\left(\frac{x^4}{4} + \frac{x^2}{2} - 5x\right)\Big|_0^2$$
$$= \frac{1}{2}\left(\frac{16}{4} + \frac{4}{2} - 10\right) - 0 = \frac{1}{2}(4 + 2 - 16) = \boxed{-2}$$

4. Consider the solid obtained by rotating the region bounded by the curves $y = 4x$ and $y = 4\sqrt{x}$ about the x-axis.
(a) (4 points) Sketch the region and label the coordinates of the intersections of the curves.



Solve

$\cancel{x} = \cancel{\sqrt{x}}$

$x = \cancel{x}$

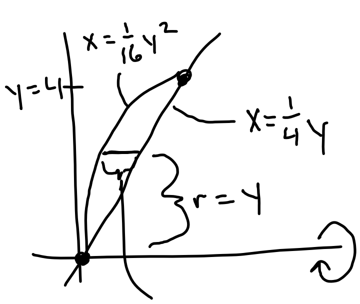
$4x = 4\sqrt{x}$

$x^2 - x = 0 \quad x(x-1)=0 \implies x = 1, x = $

(b) (8 points) Calculate the volume of the solid using washers.

$$V = \int_0^1 \pi[(4\sqrt{x})^2 - (4x)^2]\,dx$$
$$= \int_0^1 \pi(16x - 16x^2)\,dx$$
$$= 16\pi\int_0^1 x - x^2\,dx$$
$$= 16\pi\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1$$
$$= 16\pi\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{16\pi}{6} = \boxed{\frac{8\pi}{3}}$$

(c) (8 points) Calculate the volume of the solid using shells.

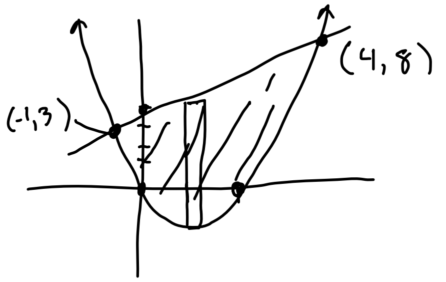


height $= \frac{1}{4}y - \frac{1}{16}y^2$

$$V = \int_{y=0}^{y=4} 2\pi(\text{radius})(\text{height})\,dy$$
$$= \int_0^4 2\pi y\left(\frac{1}{4}y - \frac{1}{16}y^2\right)dy$$
$$= \pi\int_0^4 \frac{1}{2}y^2 - \frac{1}{8}y^3\,dy$$
$$= \pi\left(\frac{1}{6}y^3 - \frac{1}{32}y^4\right)\Big|_0^4$$
$$= \pi\left(\frac{64}{6} - \frac{256}{32}\right)$$
$$= \pi\left(\frac{32}{3} - \frac{32}{4}\right) = 32\pi\left(\frac{1}{3} - \frac{1}{4}\right)$$
$$= 32\pi\left(\frac{1}{12}\right)$$
$$= \frac{32\pi}{12} = \boxed{\frac{8\pi}{3}}$$

5. Consider the region bounded by the curves $y = x^2 - 2x$ and $y = x + 4$.

(a) (4 points) Sketch the region and label the coordinates of the intersections of the curves.



Solve $x^2 - 2x = x + 4$

$x^2 - 3x - 4 = 0$

$(x-4)(x+1) = 0$

$x = 4, \ x = -1$

(b) (6 points) Set up and evaluate the integral for area of the region.

$$A = \int_{x=-1}^{x=4} (x+4) - (x^2 - 2x)\,dx$$

$$= \int_{-1}^{4} -x^2 + 3x + 4\,dx$$

$$= \left(-\frac{x^3}{3} + \frac{3}{2}x^2 + 4x\right)\Big|_{-1}^{4}$$

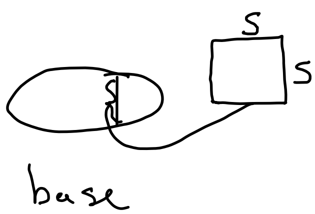
$$= \left(-\frac{64}{3} + \frac{48}{2} + 16\right) - \left(\frac{1}{3} + \frac{3}{2} - 4\right)$$

$$= -\frac{65}{3} + \frac{45}{2} + 20$$

$$= -\frac{130}{6} + \frac{135}{6} + \frac{120}{6}$$

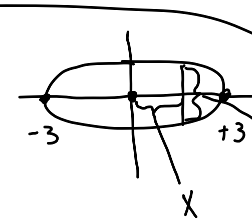
$$= \boxed{\frac{125}{6}}$$

6. (15 points) Let S be a solid with a circular base of radius 3 feet, with parallel cross-sections perpendicular to the base being squares. Set up (do not evaluate) the integral representing the volume of the solid.



$A(s) = s^2$

$= 4y^2$



$2y = s$

Formula for circle

$x^2 + y^2 = 9$

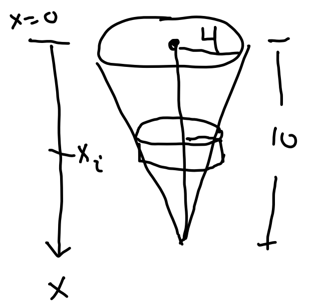
$y^2 = 9 - x^2$

So, $A(x) = 4(9 - x^2)$

$$V = \int_{-3}^{3} A(x)\,dx = \int_{-3}^{3} 4(9 - x^2)\,dx$$

$$= \boxed{8\int_0^3 (9 - x^2)\,dx}$$

7. (15 points) A tank full of water has the shape of an inverted right circular cone with base radius 4 meters and height 10 meters. Set up (do not evaluate) the integral representing the total work required to pump all of the water out of the top of the tank. (Assume that water has a density of 1000 kg/m^3)



① Volume of Slab $= \pi\left(\frac{2}{5}(10-x_i)\right)^2 \Delta x$

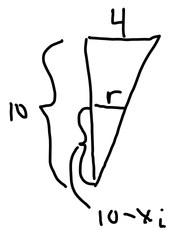
② Mass of Slab $= 1000\pi\left(\frac{4}{25}\right)(10-x_i)^2 \Delta x$

③ Force on Slab $= (9.8)(1000)\left(\frac{4\pi}{25}\right)(10-x_i)^2 \Delta x$

④ Work required to lift slab $= (9.8)(1000)\left(\frac{4\pi}{25}\right)(10-x_i)^2 \cdot x_i \Delta x$

⑤ Total work (approx) $\approx \sum_{i=1}^{n} (9.8)(1000)\left(\frac{4\pi}{25}\right)(10-x_i)^2 x_i \Delta x$

⑥ Total work $= \boxed{\int_0^{10} (9.8)(1000)\left(\frac{4\pi}{25}\right)(10-x)^2 \cdot x\,dx}$



$\frac{4}{10} = \frac{r}{10-x_i}$

$r = \frac{2}{5}(10-x_i)$

8. Evaluate each of the following integrals:

(a) (8 points) $\int \frac{1+x}{1+x^2}\,dx$

$= \int \frac{1}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx$

$= \tan^{-1}x + \int \frac{x}{1+x^2}\,dx$

Let $u = 1+x^2$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$= \tan^{-1}x + \frac{1}{2}\int \frac{du}{u}$

$= \tan^{-1}x + \frac{1}{2}\ln|u| + C$

$\boxed{= \tan^{-1}x + \frac{1}{2}\ln|1+x^2| + C}$

(b) (7 points) $\int \sec x(\tan x - \cos x)\,dx$

$= \int \sec x\tan x - \sec x\cos x\,dx$

$= \int \sec x \tan x - 1\,dx$

$= \boxed{\sec x - x + C}$