Appendix 1

Binomial distribution; HW page 726: 1, 2, 5, 8, 12.

- **Free throws**

3 attempts

On average, Big V makes $\pi = 2/3$ of her free throws. Our text uses π to represent population proportions, not $\pi = 3.14$. Remember that Greek letters apply to populations -- $\pi = \frac{2}{3}$ is the proportion of the theoretical *population* of all possible free throws Big V might attempt.

Question 1: If V takes 3 shots what's the probability that she makes the first two and misses the last?

We want to compute $P(\text{makes first AND makes second AND misses third})$. This is an AND event and so we'd like to use the multiplication rule. Can we?

--We assume that the attempts are independent -- no "hot hand".

$P(\text{makes first AND makes second AND misses third})$
$= P(\text{makes first})\, P(\text{makes second})\, P(\text{misses third})$

$$= \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(1-\frac{2}{3}\right) = \frac{4}{27} = 0.148148$$

Question 2: Big V shoots 3 and we let x be the number of free throws she makes.

What is $P(x = 0)$?   --misses first AND misses second AND misses third

What is $P(x = 3)$?   --makes first AND makes second AND makes third

What is $P(x = 2)$? S-stands for success, F-for fail.

There are three ways that V could make 2 out of 3   -- SSF OR   SFS OR FSS

$P(x = 2) = P(\text{SSF or SFS or FSS})$ This is an OR statement. Can we use the sum rule?

$$= P(\text{SSF}) + P(\text{SFS}) + P(\text{FSS}) = \frac{4}{27} + \frac{4}{27} + \frac{4}{27} = \frac{12}{27} = 0.444444$$

NOTE how each of the three possible ways had the same probability of $\frac{4}{27}$.

4 attempts

What's the probability that V makes two out of four?

-- first, how many ways can this happen?   -- second, what's the probability of any one? -- third, multiply.

Here are all of the possible 16 outcomes from taking 4 shots.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Shot 1:	S	S	S	S	F	S	S	S	F	F	F	S	F	F	F	F
Shot 2:	S	S	S	F	S	S	F	F	S	S	F	F	S	F	F	F
Shot 3:	S	S	F	S	S	F	S	F	S	F	S	F	F	S	F	F
Shot 4:	S	F	S	S	S	F	F	S	F	S	S	F	F	F	S	F

There are six orderings for making two of four shots.

$P(x = 2) = P(\text{SSFF or SFSF or SFFS or FSSF or FSFS or FFSS}) =$

9 attempts

What's the probability that V makes four out of nine?

There are $2^9 = 512$ possible S/F outcomes for 9 shots--too many to write out.

The number of ways to choose k things from n is denoted by the *binomial coefficient* $\binom{n}{k}$--read aloud as "n choose k". Your calculator might compute binomial coefficients. If not, you use this formula: $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$.

We want to know the number of ways to choose which 4 of the nine shots V will make.

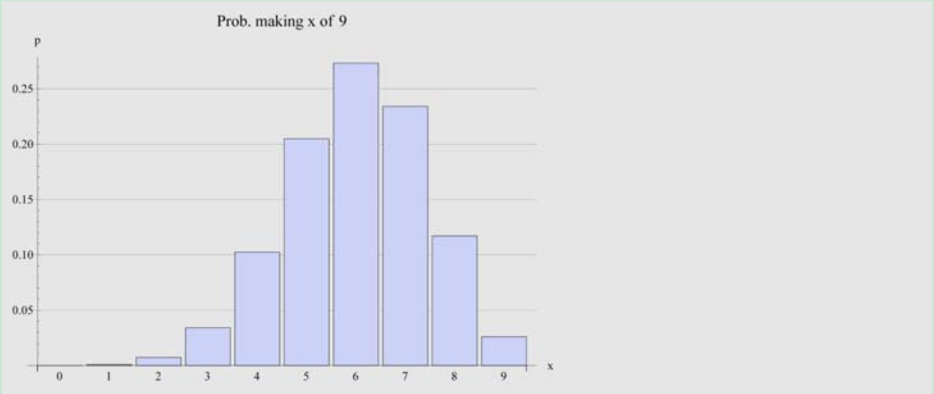
The answer is

$$\binom{9}{4} = \frac{9!}{4!\,(9-4)!} = \frac{9!}{4!\;\;5!} = \frac{(9)(8)(7)(6)(5)(4)(3)(2)(1)}{(4)(3)(2)(1)\;\;(5)(4)(3)(2)(1)} = \frac{(9)(8)(7)(6)}{(4)(3)(2)(1)} = \frac{(9)(7)(6)}{(3)(1)} = (3)(7)(6) = 126.$$

The probability of any one of these 126 ordering is the same as $P(\text{SSSSFFFFF}) = \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^5 = \frac{16}{19683} = 0.000812884$.

So, $P(x = 4) = \binom{9}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^5 = (126)\,\frac{16}{19683} = 0.102423$.

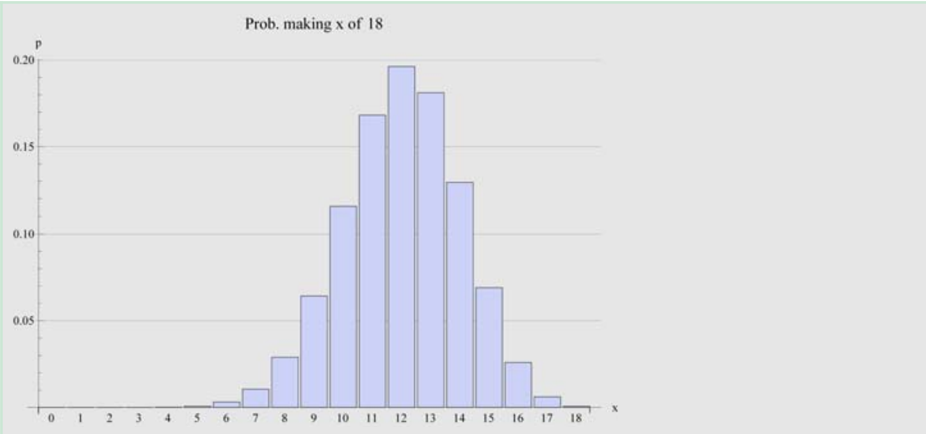
Histogram for 9



Describe the distribution and estimate the mean.

From the histogram it looks like the probability that V makes 4 of 9 is a little more than $\frac{1}{10}$. Compute the exact value. Express your answer in three ways: (a) the exact answer in terms of a binomial coefficient, (b) the exact answer without a binomial coefficient, (c) a decimal approximation.

Histogram for 18



Describe the distribution and estimate the mean.

From the histogram it looks like the probability that V makes 12 of 18 is a little less than $\frac{2}{10}$. Compute the exact value. Express your answer in three ways: (a) the exact answer in terms of a binomial coefficient, (b) the exact answer without a binomial coefficient, (c) a decimal approximation.

Rolling dice

You roll a fair six sided die. What the probability that you get a six?

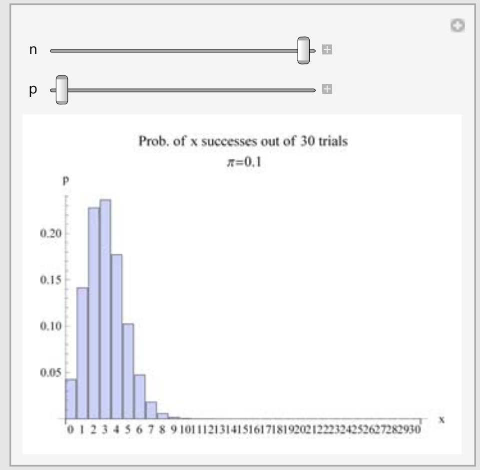
You roll a fair six sided die 100 times. What the probability that you get twenty 6s?

Binomial distribution

A binomial experiment consists of n repeated trials. Each trial ends in either a "success" or a "failure". The probability of a success on each trial is a fixed number, usually denoted by π. Since there are only two possible outcomes for a trial, the probability of a failure is $1-\pi$.

If x counts the number of successes, then x "has a binomial distribution".

Examples



The number x is not a fixed number. It is a "random variable" with a binomial distribution.

The mean for x is $\mu_x = n\pi$ (memorize)

The standard deviation for x is $\sigma_x = \sqrt{n\pi(1-\pi)}$. (on crib sheet)

Formula

⟶ n independent trials with success probability π

⟶ x counts the number of successes

$$P(x \text{ out of } n \text{ successes}) = p(x) = \binom{n}{x}\pi^x(1-\pi)^{n-x}$$ (memorize)

Example

Suppose that 20% of the parts produced by a certain machine are defective.

You randomly sample 50 parts from the machine. What is the probability that 10 of the parts are defective?

Number of trials $n = 50$,

Define "success = defective part". Let x be the number of defective parts.

Argue that the probability of success for each trial is $\pi = 0.2$. (Is it?)

Now we can use the formula:

$$p(10) = \binom{50}{10} 0.2^{10}\,(1-0.2)^{50-10} = (10\,272\,278\,170)\,0.2^{10}\,(0.8)^{40} = 0.139819.$$

Sampling without replacement

With

We have 10 playing cards, half are red.
We randomly choose two cards WITH replacement.
Question: What's the probability that both are red?

Answer: $\frac{1}{2}\,\frac{1}{2} = \frac{1}{4} = 0.25$

Without

We have 10 playing cards, half are red.
We randomly choose two cards WITHOUT replacement. What's the probability that both are red?

Answer: $\frac{5}{10}\,\frac{4}{9} = \frac{20}{90} = 0.2222222222222222222222222222222$

Sampling WITHOUT replacement affects the probabilty of subsequent trials.

Without

We have 1000 playing cards, half are red.
We randomly choose two cards WITHOUT replacement. What's the probability that both are red?

Answer: $\frac{500}{1000}\,\frac{499}{999} = \frac{499}{1998} = 0.24975$

Observation: If we sample only a tiny fraction of the population, REPLACEMENT $\approx$ WITHOUT REPLACEMENT

RULE of Thumb

Sampling without replacement causes dependence in the trials.
This creates error in the binomial formula.

RULE OF THUMB: If you sample without replacement you can still use the binomial formula as long as you do not sample more than 10% of the total population. (Memorize)

Example

Suppose that 60% of the 50 students in this class are from Wisconsin.
You randomly select and interview 10 of the students and ask them whether they are from Wisconsin.
What is the probability that you will find exactly 7 students from Wisconsin?
If you view this as a binomial experiment with $n = 10$, $\pi = 0.6$, then

$p(7) = \binom{10}{7}(0.6)^7\,(.4)^3 = 0.214991.$ BUT THIS IS WRONG!

To avoid annoying people you sampled the students *without replacement*.
Once a student was interviewed he or she was eliminated from the pool before drawing the next person.
What fraction of the population did you sample?

Example

Suppose that 60% of the 1000 students at a university are from Wisconsin.
You randomly select and interview 10 of the students and ask them whether they are from Wisconsin.
What is the probability that you will find 7 students from Wisconsin?
If you view this as a binomial experiment with $n = 10$, $\pi = 0.6$, then

$p(7) = \binom{10}{7}(0.6)^7\,(.4)^3 = 0.214991.$

To avoid annoying people you sampled the students *without replacement*.
What fraction of the population did you sample?

Sampling without replacement means your answer of $p(7) = \binom{10}{7}(0.6)^7\,(.4)^3 = 0.214991$ is not quite right.
But the 10% rule of thumb says that it's close enough.

(The actual probability is: $\frac{\binom{600}{7}\binom{400}{3}}{\binom{1000}{10}} = 0.215529$.)