Chapter 6

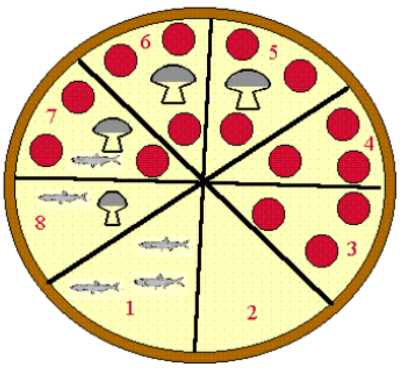
Probability: Multiplication and Sum rules
Read chapter 6; page 258: 1, 3, 11; page 263: 15, 17; page 273: 19, 21; page 277: 28

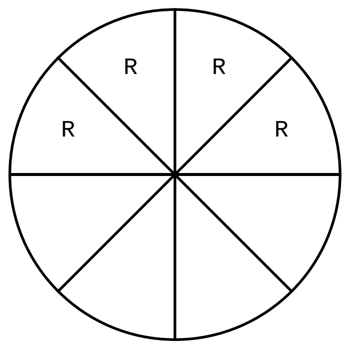
HW page 258: 1, 3, 11; page 263: 15, 17; page 273: 19, 21; page 277: 28

Probability notation: The probability that an event A happens is $P(A)$.

Example. You roll a fair six sided die and record the number as x. Consider the event "x is 2". The probability that x is a two, denoted $P(x \text{ is } 2)$ is $\frac{1}{6}$ since on a fair die each number is equally likely.

You are blindfolded and pick a slice from the Pepperoni, Anchovy, Mushroom pizza shown above on the left. Let R represent the event that the slice has Pepperoni, A represent that the slice has Anchovy, and M that the slice has Mushroom.





1. To compute $P(M)$ you count that 4 of the 8 slices have mushrooms, so $P(M) = 4/8 = 1/2$.

a. Compute $P(R)$

b. Compute $P(A)$

Complement rule: For any event A, $P(\text{not } A) = 1 - P(A)$.

Example. You roll a fair six sided die and record the number as x. Consider the event "x≠5".

$$P(x \neq 5) = 1 - P(x = 5) = 1 - \frac{1}{6} = \frac{5}{6}.$$

2. For the pizza, compute $P(\text{not } R)$ in two ways.

a. By directly counting the fraction of slices which do not have pepperoni.

b. By applying the complement rule.

Disjoint events: Events A and B which cannot happen at the same time are *disjoint events.*

Sum rule: If A and B are disjoint, then $P(A \text{ OR } B) = P(A) + P(B)$.

Example.You roll a fair six sided die and record the number as x. The event "x is 2" and the event "x is 3" are *disjoint* since x can't be both 2 and 3 at the same time. Since the events are disjoint you use the sum rule to compute $P(x \text{ is } 2 \text{ OR } x \text{ is } 3) = P(x \text{ is } 2) + P(x \text{ is } 3) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$.

3. Compute $P(R \text{ or } M)$ by counting the fraction of the slices that have either pepperoni or mushroom *or both*.

4. Compute $P(R)+P(M)$.

5. The sum rule says $P(R \text{ or } M) = P(R)+P(M)$. Why didn't this turn out to be the case for your answer to (3) and (4)? Explain using the word "disjoint".

6. The pizza above on the right has four slices with pepperoni. Add mushrooms (M) to two slices of the pizza in such a way that the events R and M are disjoint. Test your answer by checking if $P(M \text{ or } R) = P(M)+P(R)$.

Independent events: Events A and B are independent if A happening doesn't influence whether B happens (and vice versa). Although in everyday language "independent" might seem like a synonym for "disjoint" the terms *disjoint events* and *independent events* mean different things.

Multiplication rule: If A and B are independent events, then $P(A \text{ AND } B) = P(A)\,P(B)$.

Example: You roll a die and record the number as x. You roll again and record the number as y. Compute $P(x \text{ is } 2 \text{ AND } y \text{ is } 3)$. The value of x on the first roll doesn't influence the value for y on the second roll so the events "x is 2" and "y is 3" are independent. So you can use the multiplication rule to compute: $P(x \text{ is } 2 \text{ AND } y \text{ is } 3) = P(x \text{ is } 2)\,P(y \text{ is } 3) = \frac{1}{6}\frac{1}{6} = \frac{1}{36}$.

7. Compute $P(R \text{ and } M)$ by counting the fraction of the slices have both pepperoni and mushroom.

8. Compute $P(R)\,P(M)$.

9. The multiplication rule says that $P(R \text{ and } M) = P(R)\,P(M)$. Why didn't this turn out to be the case for your answers to (7) and (8)? Answer using the new term you have just learned.

10. Remembering that you are blindfolded when you pick your slice, if you smell anchovy on your slice, what is the probability that the slice has pepperoni. The notation for this probability is $P(R \mid A)$. Compute $P(R \mid A)$ by counting the fraction of the anchovy slices that have pepperoni on them. Compare $P(R)$ and $P(R \mid A)$. You see that the event "has anchovies" influences the probability of "has pepperoni". So R and A are dependent events.

11. Events A and B are independent if any of the following three equations are true: $P(A \text{ and } B) = P(A)\,P(B)$, $P(A \mid B) = P(A)$, $P(B \mid A) = P(B)$. If any of the equations are false, then A and B are dependent. Determine whether A and M independent by checking any one of the three equations.

12. The pizza on above on the right has four slices with pepperoni. Add mushrooms (M) slices to two slices of the pizza in such a way that the events R and M are independent. Hint: Try to arrange for $P(M) = P(M \mid R)$

Computing probabilties

Computing probabilities by direct count

Choose a random card from among these six:

5 ♠, 6♠, 7♠, 3♢, 8♢, 9♢

$P(\text{black})$, $P(\text{even})$, $P(\text{prime})$

$P(\text{black OR even})$, $P(\text{black OR prime})$

$P(\text{black AND even})$, $P(\text{black AND prime})$

$P(\text{even} \mid \text{black})$, $P(\text{black} \mid \text{even})$, $P(\text{black} \mid \text{prime})$, $P(\text{prime} \mid \text{black})$

Dice

Roll two dice and let x be the sum of the pips.

Possible values for x are: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.

$P(x = 2)$? $P(x = 7)$?

Here are the possible EQUALLY LIKELY outcomes from rolloing the TWO dice.

```
{{1, 1}, {1, 2}, {1, 3}, {1, 4}, {1, 5}, {1, 6}}
{{2, 1}, {2, 2}, {2, 3}, {2, 4}, {2, 5}, {2, 6}}
{{3, 1}, {3, 2}, {3, 3}, {3, 4}, {3, 5}, {3, 6}}
{{4, 1}, {4, 2}, {4, 3}, {4, 4}, {4, 5}, {4, 6}}
{{5, 1}, {5, 2}, {5, 3}, {5, 4}, {5, 5}, {5, 6}}
{{6, 1}, {6, 2}, {6, 3}, {6, 4}, {6, 5}, {6, 6}}
```

How many of these 36 outcomes have a sum of 2? ...a sum of 7?

$P(x = 2)$

$P(x = 7)$

$P(x \leq 4)$?

$P(x > 4)$?

$P(x \text{ even} \mid x \leq 5)$?

Estimating probabilities by simulation

Suppose you roll 100 dice and let x be the sum of the pips.

$P(x = 350)$?

For two dice there were $6^2 = 36$ possible outcomes.

For 100 dice there are $6^{100} =$ possible outcomes.

```
6^100
```

```
653 318 623 500 070 906 096 690 267 158 057 820 537 143 710 472 954 871 543 071 966 369 497 141 477 376
```

```
RandomInteger[{1, 6}, 100]
```

```
{1, 3, 1, 2, 3, 1, 5, 1, 2, 1, 1, 3, 1, 1, 2, 1, 6, 4, 6, 6, 5, 4, 2, 3, 3, 5, 5, 3, 3, 3, 2, 3,
 5, 6, 4, 1, 3, 6, 3, 4, 4, 6, 4, 3, 5, 5, 2, 4, 6, 6, 1, 4, 4, 3, 4, 5, 2, 5, 2, 4, 5, 3, 1, 5, 4, 2,
 2, 3, 6, 6, 1, 5, 3, 2, 2, 5, 1, 2, 3, 5, 1, 6, 5, 1, 4, 3, 2, 1, 5, 6, 6, 6, 6, 5, 6, 6, 1, 4, 4, 1}
```

```
Total[RandomInteger[{1, 6}, 100]]
```

```
332
```

```
Table[Total[RandomInteger[{1, 6}, 100]], {25}]
```

```
{373, 330, 338, 370, 354, 346, 381, 370, 326, 348, 354,
 360, 351, 327, 306, 342, 337, 359, 363, 348, 335, 355, 366, 318, 384}
```

```
Count[Table[Total[RandomInteger[{1, 6}, 100]], {1000000}], 350]
```

```
23311
```

Estimated probability of getting 350 as a sum.

```
23141 / 1000000.
```

```
0.023141
```

```
Histogram[Table[Total[RandomInteger[{1, 6}, 100]], {100000}]]
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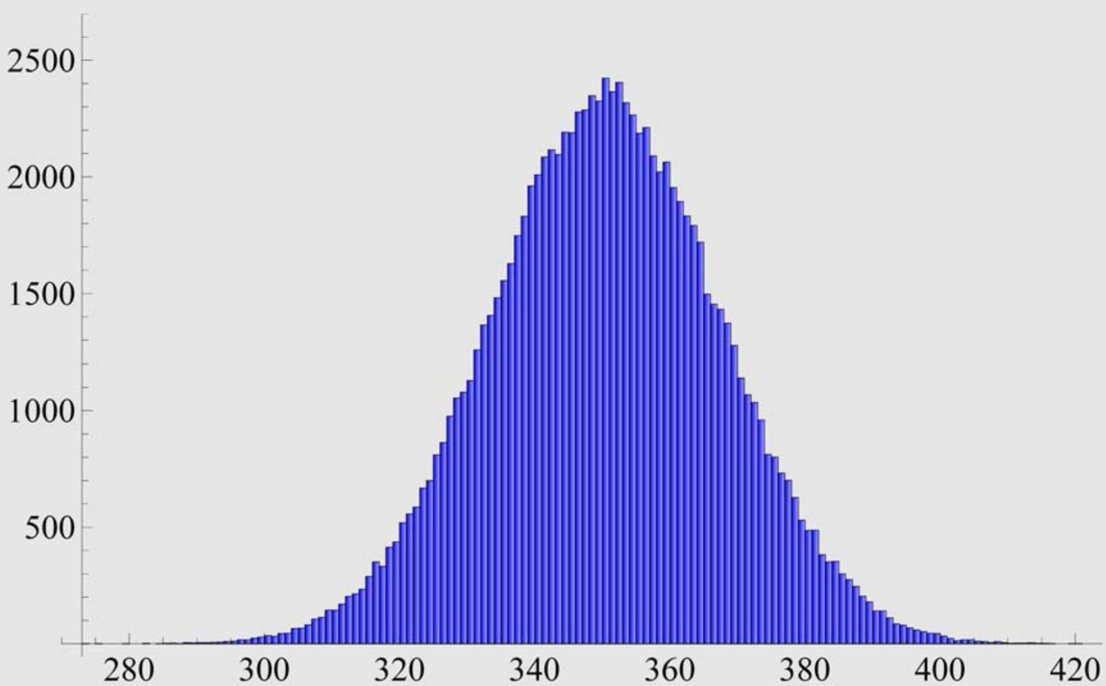

- **Estimating probabilities by long run experience**

Chance of snow is 30%.

Independence, conditional probability

Definition

Events A and B are independent if the chance of one event is NOT affected by knowledge of whether or not the other event occurs.

Judging independence by intuition

Independent

You pick a random date from the year 2008.
Event R is "it rained". Event M is that the date was a Monday.
Our intuition is that these events are INDEPENDENT--The day of the week doesn't affect whether it rains.

Dependent

Choose a random vehicle from the parking lot.
Let H be "has a hitch". Let T be "is a truck".
Our intutition is that these events are DEPENDENT--Trucks are more likely to have hitches than cars.

Judging independence by calculations

Choose a random card from among these six:

5 ♣, 6♠, 7♠, 3♢, 8♢, 9♢

Are R red and E even independent? What about red and prime?

Venn diagrams

Choose a student at random from a class of 100 students.
Let M be "has a MasterCard", let V be "has a Visa Card".
What's your intutition?

Suppose 60 students have a MasterCard, 70 students have a Visa card and 40 students have both cards.

Suppose 60 students have a MasterCard, 70 students have a Visa card. If the events M and V are independent, how many students have both cards?

Contigency table

In a class of 40 men and 60 women there are 50 democrats, 30 republicans, and 20 socialists.
If M male and S socialist are independent events, how many men must be socialists?
If F female and R republican are independent events, how many women must be republicans?