
Introduction

● What are discrete methods?

♣ Example - Continuous variables:

Show that there is no real number x such that $\sin(x^2 + 1) = 5 + \cos(x)$.

Can't check all x 's. Need to be clever.

For any x , $\sin(x^2 + 1) \leq 1$ and $4 \leq 5 + \cos(x)$, so two can't be equal.

♣ Example - Discrete Problem:

Show no collection of five numbers from $S = \{18, 33, 102, 141, 261, 282, 312, 342, 360, 402, 447\}$ sum to 1000.

There are only a finite number of possible sets of five that you could try. In fact, if we allow repeated numbers, there are $11^5 = 161\,051$ possible sums. We could check all possibilities. This is common to discrete problems. But discrete methods are the clever, often beautiful methods and observations that allow us to avoid checking all possibilities.

Every number from S is a multiple of 3, so any sum will be a multiple of 3. But 1000 is not a multiple of 3, so it is not possible to choose 5 numbers (or any numbers) from S summing to 1000.

1.1 Graph Models

HW: 2, 4, 6, 7(explain), 18, 20, 24(prove), 30, 32, 36

● Graph definition and language

Definition: A graph G with n vertices and m edges consists of a vertex set $V(G) = \{v_1, \dots, v_n\}$ and an edge set $E(G) = \{e_1, \dots, e_m\}$, where each edge is an unordered pair of vertices.

Example: $V(G) = \{a, b, c, d\}$ and $E(G) = \{(a, b), (a, c), (b, d), (c, d), (a, d)\}$.

Draw the picture (square with diagonal). Curved lines allowed. Different arrangements of the vertices allowed.

Alternate notation for edges: (a, b) , ab , $a \leftrightarrow b$

● Language

Any set can be a "vertex" set. (infinite sets allowed too).
Vertex, index \rightarrow vertices, indices.

Two vertices u, v are **adjacent** if (u, v) is an edge. (if there is an edge between them)
E.g. $_$ and $_$ are non-adjacent.

The vertices adjacent to v are called the **neighbors** of v .
E.g. $\{a, d\}$ are the neighbors of b .

The **degree of v** , $deg(v)$, $d(v)$ is the number of neighbors of v .

The edge (u, v) is **incident** to u and incident to v . Each edge is incident to two vertices, namely its endpoints. A vertex is incident to all edges that use it as an endpoint.

E.g. vertex b is incident to $_$

A **path** is a sequence of distinct vertices (u_1, \dots, u_n) where consecutive vertices are adjacent.

E.g. (b, a, d, c) is a path

E.g. (b, c, d) is not a path.

E.g. (a, b, d, a, c) is not a path.

The path (u_1, \dots, u_n) has **length** $n-1$. (The number of edges.)

E.g. (c, d, b) is a length 2 path.

A **circuit** is a path which is allowed to finish where it starts.

E.g. (a, d, c, a) is a length 3 circuit. (sometimes called a cycle.)

A **clique** is a set of mutually adjacent vertices.

E.g. $\{a, b, d\}$ is a clique of size 3.

An **independent set** is a set of mutually non-adjacent vertices.

E.g. $\{c, b\}$ is an independent set of size 2.

A graph is **connected** if there is a path between every pair of vertices.

(Vertices are adjacent, graphs are connected.)

■ Non-simple graphs.

Directed graphs (digraph), multigraphs, loops.

● Graph models

■ Ramsey Numbers

Every group of 6 people has three mutual friends or three mutual strangers.

Model with a "friendship" graph. Let $V = \{1, 2, 3, 4, 5, 6\}$ represent the 6 people. Two vertices are adjacent if and only if the two people are friends.

(Typical way to describe a graph. Specify the vertex set, give a rule to say when two vertices are adjacent.)

☞ Show that every graph on 6 vertices has either a size 3 clique or a size 3 independent set.

Proof.

Case 1: Suppose $\deg(1) \geq 3$. If two neighbors of vertex 1 are adjacent, then they, along with vertex 1 form a clique of size 3. On the other hand, if no two neighbors of vertex 1 are adjacent, then the neighbors themselves form an independent set of size at least 3.

Case 2: Suppose $\deg(1) < 3$. If two non-neighbors of vertex 1 are non-adjacent, then they, along with vertex 1, form an independent set of size 3. On the other hand, since vertex 1 has at least 3 non-neighbors and if no two of them are non-adjacent, then these non-neighbors form a clique of at least size 3.

Comments: Typical case structure for proof. Typically "few" calculations.

Current research on Ramsey numbers.

■ Matching

Each of 5 kids a, b, c, d, e will get one of six pieces of candy 1, 2, 3, 4, 5, 6

However, each kid submits a list of candy he/she is willing to accept.

$a \rightarrow 4, 6$

$b \rightarrow 1, 3, 5, 6$

$c \rightarrow 2, 4, 6$

$d \rightarrow 2, 4$

$e \rightarrow 2, 6$

Draw a "preference" graph: let $V = \{a, b, c, d, e\} \cup \{1, 2, 3, 4, 5, 6\}$ and (x, n) will be an edge if and only if kid x will accept candy n. This is an example of a **bipartite graph**.

☞ Is it possible to please all of the kids?

Answer: $\{a, c, d, e\}$ together will accept only $\{2, 4, 6\}$.

■ Surveillance - Example 4 page 6.

Cops to be assigned to corners to watch blocks.

There must be a cop at least one end of every block.

First, just compute minimum number of cops required.

Common format for "find minimum" problems.

Demonstrate n, argue that n-1 is not possible.

Text proves that there is only one edge cover of size 5.

Edge Covers

Definition: A set of vertices C is an edge cover if every edge is incident to at least one vertex in C (or if every edge has at least one endpoint in C).

☞ **Theorem:** Suppose that I and C partition V. Then I is an independent set if and only if C is an edge cover.

Proof:

If C is an edge cover, then each edge has at least one endpoint in C. So no edge has both endpoints in I. So no two vertices in I are adjacent. So I is an independent set.

If I is an independent set, then no edge has both endpoints in I. So each edge has at least one endpoint in C. So C is an edge cover.

■ Edge weighted graphs.

- MST
- TSP
- Network Flow

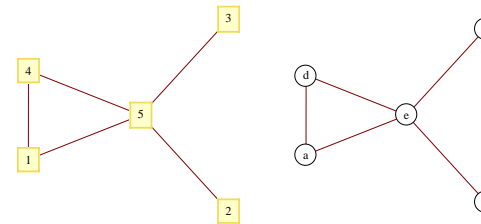
1.2 Isomorphism

HW: 1.2: Problem 6, 11, 12(justify), 16.

Definition: two graphs G and H are isomorphic if there exists a one to one correspondence $f: V(G) \rightarrow V(H)$, which preserves adjacencies, i.e. For a, b in $V(G)$, (a, b) is an edge in G iff $(f(a), f(b))$ is an edge in H.

■ Example

These graphs are isomorphic



To show that two graphs are isomorphic you need to give the correspondence.

More often the isomorphism is hard to spot since the graphs are not drawn the same.

■ **Example: 5a page 21.**

Informally, two graphs are isomorphic if one can be drawn like the other.
Imagine dragging one graph on top of the other.
Give the correspondence/isomorphism

■ **Structural Features**

If two graphs are isomorphic, (you can draw one on top of the other), then they must have identical structural features.
E.g. the same

- number of vertices
- number of edges
- circuits and paths: If G has a path of length 4 so does H.
- degree sequence: a sorted list of the degrees
- subgraphs: a graph specified by a subset of the edges
- induced subgraph: graph specified by a subset of the vertices
- complements: the complement of $G = (V, E)$ is $\bar{G} = (V, \bar{E})$, where \bar{E} denotes the complementary set of edges.

Key point: if graphs have many edges, perhaps study complements

Key point: induced subgraphs on sets of specified degrees must be isomorphic (Do Example 1)

■ **How do you show G and H are non-isomorphic?**

Find a structural difference.

■ **How do you show G and H are isomorphic?**

Look for features in one that must be matched with the other.

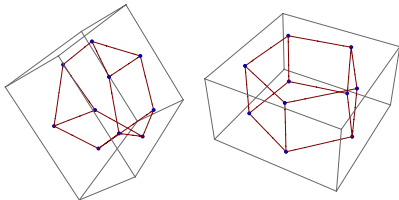
Extend these matchings to an isomorphism/correspondence.

Give the correspondence--explanation of how you found correspondence is not required.

🔗 **Pair up and do 5b-5f.**

■ **Problem 6g**

Problem 6g



■ **Complexity of Graph Isomorphism.**

1.3 Edge Counting

HW: 2, 4, 6, 12

Practice the two extremal proofs from this section.

■ **New proof technique: Counting the same set in two different ways.**

Theorem: For any graph G, the sum of all the vertex degrees equals twice the number of edges.

Proof: Let $S = \{(v, e) : v \text{ is incident to } e\}$.

Count 1: Each edge contributes two elements to S, so $|S|=2|E|$.

In symbols $|S| = \sum_{e \in E(G)} 2 = 2|E|$.

Count 2: Each vertex contributes its degree to the size of S, so $|S|$ equals the sum of all the degrees. In sym-

bolts $|S| = \sum_{v \in V(G)} \text{deg}(v)$.

Ergo, $\sum_{v \in V(G)} \text{deg}(v) = 2|E|$.

Corollary: The number of edges in a graph is half the sum of all the degrees.

🔗 How many edges in a 3-regular (all vertices with degree 3) graph on 12 vertices?

🔗 How many edges in a 3-regular graph on 11 vertices?

🔗 In any graph what must be true about the number of vertices with odd degree?

🔗 How many edges in the complete graph on n vertices, K_n ?

■ **New proof technique: Extremal arguments.**

Theorem: If every vertex in a graph G has degree two or greater, then G has a circuit.

Awkward proof: Start at vertex v_0 . Since the degree is at least two, v_0 has a neighbor v_1 . Since $d(v_1) \geq 2$, v_1 has some neighbor in addition to v_0 , say v_2 . Again v_2 has some neighbor in addition to v_1 either its v_0 in which case we have a circuit or its another vertex v_3 , etc.....

In this proof "..." refers to the possibility of ever longer paths.

We simplify the proof by taking the **EXTREME**.

Proof: Let $P = (v_0, \dots, v_k)$ be a **longest** path in G.

Since P is a longest path, all of v_0 neighbors are on the path P .

(If v_0 had a neighbor v not on P , then (v, v_0, \dots, v_k) would be a longer path.)

Since $\text{deg}(v_0) \geq 2$, v_0 has at least two neighbors on P and so at least one neighbor v_i with $i \geq 2$. So (v_0, \dots, v_i, v_0) is a circuit of length $i \geq 2$.

■ Another Extremal argument.

Definition: Parity is even or odd. Two paths have opposite parity if one has even length and the other has odd length.

Theorem: If G has opposite parity paths between a pair of vertices, then G has an odd length circuit.

Awkward Proof: Suppose G has opposite parity paths P and Q between vertices u and v . Then travelling from u to v along P and then back to u along Q forms an odd length circuit, unless P and Q intersect at some other vertex z . But if P and Q intersect at z then there will be shorter opposite paths from z to either u or v . Repeat the argument on these shorter paths.....

☞ First line of extremely proof?

Proof: Choose the opposite parity paths P, Q between a pair of vertices so that the sum of the lengths of P and Q is minimum. The paths can intersect only at the endpoints otherwise they would be a pair of opposite parity paths with shorter lengths. Combining P and Q forms an odd length circuit.

■ Characterization of bipartite graphs

Theorem: A graph G is bipartite iff G has no odd length circuits.

Proof: → Assume G is bipartite.

This means that the vertices can be partitioned into independent sets A and B . Vertices along any path alternate between A and B . Thus any path starting and ending in the same set has even length. In particular any circuit must have even length.

← Assume that G has no odd length circuit.

To show that G is bipartite we'll construct a partition into two independent sets A and B .

Construction: Choose any vertex x and place in A . For each vertex y , put y in A if there's an even length path from x . Put y in B if there's an odd length path from x . Well defined: There can't be a path of each type from x to y . (WHY?)

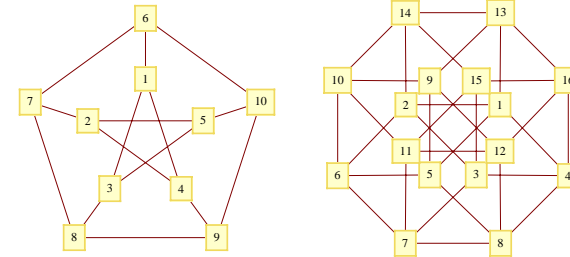
Show that each of A and B is an independent set. Suppose (u, v) is an edge in B . Since u is in B there is an odd length path from x to u . But adding the edge (u, v) to this path gives an even length path to v . (WHY?).

Similarly for A .

How do you determine if G is bipartite?

Key point: Determine the two independent sets using the construction in the proof OR give an odd length circuit.

```
GraphicsArray[{GraphData["PetersenGraph"], GraphData[Last[GraphData["Bipartite", 16]]]}]
```



HW comments

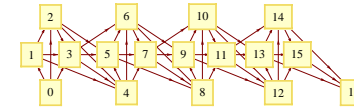
Section 1.1: 7

Section 1.1: 20--give analysis showing why you've got all sets.

Section 1.1: 24--give 3, prove 2 is not possible.

Section 1.1: 30 Explain why unit is not possible.

Section 1.1: 36



☞ **Theorem:** Suppose that I and C partition V . Then I is an independent set if and only if C is an edge cover.

Proof: Suppose I is an independent set. Then no edge has both endpoints in I . So every edge has at least one endpoint in C . So C is an edge cover.

Suppose C is an edge cover. Then every ..

OR

By definition I is an independent set if no edge has both endpoints in I .

Having no edge with both endpoints in I is the same as every edge having at least one endpoint in C .

This is the definition of C being an edge cover.

1.4 Planar Graphs

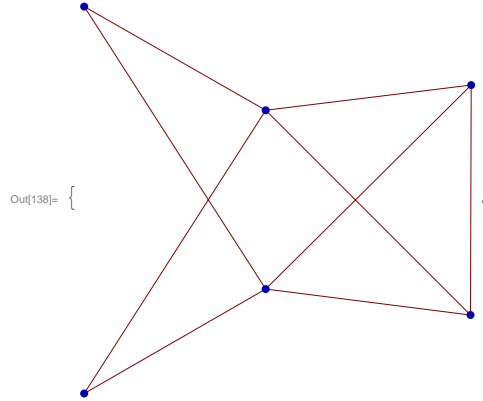
Do: 3, 5, 7

Hand in: 2, 4, 8, 13 b, 14, 16, 18 a

Planar Graphs

Definition: A graph is planar if can be drawn in the plane without crossing edges.

```
In[137]:= Clear[a, b, c, d, e, f]
GraphEdit[{a -> b, a -> c, a -> e, a -> f, b -> d, c -> d, c -> f, d -> e, d -> f}]
```



Out[138]= {

```
Graph -> {1 -> 2, 1 -> 3, 1 -> 4, 1 -> 5, 2 -> 6, 3 -> 6, 3 -> 5, 6 -> 4, 6 -> 5},
Coordinates -> {{283, 628}, {24, 224}, {576, 664}, {24, 776}, {575, 336}, {283, 373}},
VertexLabels -> {a, b, c, e, f, d}}
```

- 3c -- find a plane drawing.
- K_4 is planar even though you might not think so.
- K_5 and $K_{3,3}$ are non-planar.
- K_5 and $K_{3,3}$ minors (read configurations) are non-planar.
- Theorem:** A graph G is non-planar iff it contains a K_5 or $K_{3,3}$ minor.
- Proof:** West's book.

■ **Circle-chord method**

The circle-chord method for drawing a planar graph.

Find a circuit containing all of the vertices. Draw this as a circle. Connect other edges inside or outside the circle. It's not critical whether you put the first chord inside or outside as topologically these are the same kinds of regions. If you end up where you have to cross chords, this will produce a $K_{3,3}$ minor or K_5 minor which will prove the graph is non-planar.

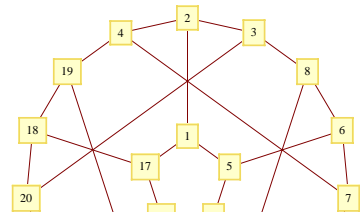
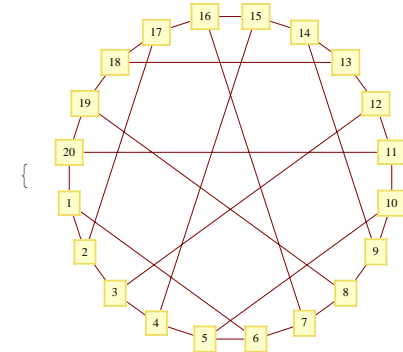
The circle-chord method for finding a $K_{3,3}$ minor.

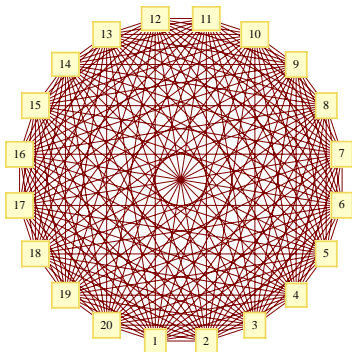
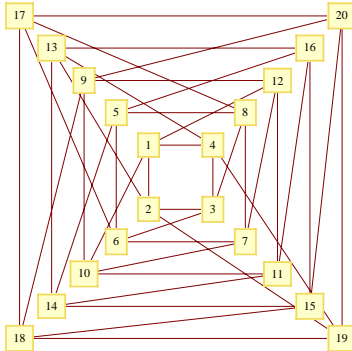
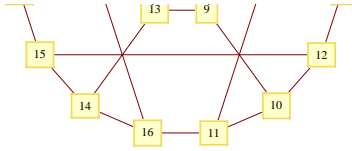
- Find a long circuit.
- Look for three crossing chords-perhaps passing through vertices not on the circuit.
- The ends of the chords form the six vertices of the minor.
- Draw the minor.

Example: 3cg (NOTE 3c doesn't work with circle-chord method).

■ **Euler's Formula**

```
GraphData /@ Complement[GraphData[20], GraphData["Planar", 20]][{1, 2, 3, 5}]
```





The more edges you have the harder it is not to have a crossing.

Euler's Formula: If G is a connected planar graph with v vertices and e edges, then any plane drawing has $r = e - v + 2$ regions.

Proof:

Base case: Suppose $e=1$. Then $v=2$ and in any plane drawing of G $r=1$, so the formula holds.

Induction Step: Suppose $e > 1$.

Let H be a plane drawing of G and suppose H has r regions.

Case 1) H has a vertex v with $\deg(v)=1$.

Form H' by deleting v . Since $\deg(v)=1$, H' is still connected.

H' is a plane graph drawing of a connected planar graph with $e-1$ edges and $v-1$ vertices.

By the induction hypothesis, $r' = (e-1) - (v-1) + 2$. Thus, $r' = e - v + 2$.

But $r=r'$.

Case 2) Every vertex of H has degree 2 or greater.

Then by previous theorem, we know that H has a circuit.

Let C be a shortest circuit, and let (x,y) be an edge on C .

Form H' by deleting the edge (x,y) and let r' be the number of regions in H' .

Since (x,y) is on a circuit, H' is still connected.

H' is a plane graph drawing of a connected planar graph with $e-1$ edges and v vertices.

By the induction hypothesis, $r' = (e-1) - (v) + 2$. Thus, $r' = e - v + 1$.

Since C is the shortest circuit, it has no chords.

Removing (x,y) joins the inside of the circuit with the outside of the circuit, reducing the number of regions by one.

Thus H' has $r-1$ regions.

$(r-1) = r' = (e-1) - (v) + 2$. Adding 1 to both sides gives the desired result.

■ **Edge bound for planar graphs**

Definition: the degree of a region is the number of edges along its boundary. (Some edges may be counted twice, give example with a leaf in the interior of the region.)

Corollary: If G is a connected planar graph with $e \geq 2$, then $e \leq 3v - 6$.

Proof:

Observation: The sum of the degrees of all regions is $2e$.

Observation: If G has two or more edges, then the degree of each region is at least 3.

Observation: The sum of the degrees of all the regions is at least $3r$.

With these three observations we know $3r \geq 2e$. Substituting the result of the previous Theorem we get $3(e - v + 2) \geq 2e$.

Solving for e yields the result.

Contrapositive: If G is connected with $v \geq 3$ and $e > 3v - 6$, then G is not planar.

Warning: the converse is not true: $e \leq 3v - 6$ DOES NOT IMPLY planar.

Example $K_{3,3}$

Give example where $e \leq \frac{1}{100}v$ and yet still non-planar.

K_5 plus 995 isolated vertices.

■ **Is G planar?**

Yes, here's the drawing.

No, EITHER it is connected and has too many edges OR here's the K_5 or $K_{3,3}$ minor.

■ **Review**

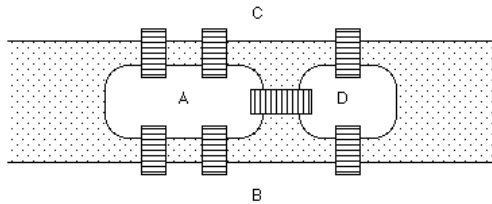
- ❖ Is G bipartite?
- ❖ Are G and H isomorphic?
- ❖ Find the minimum size of an edge cover.
- ❖ Find the maximum size independent set.

2.1 Euler Cycles.

2.1: 2, 8, 10, 12a, 16, 19d

□ **Konigsberg bridge problem.**

Spaziergangen. Cross all bridges once and return to start.
(Hey they didn't have TV.)



What do you think? Try it.

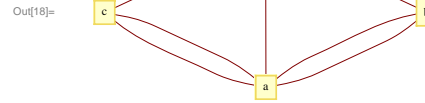
□ **Graph Model.**

Vertex for each land mass, edge for each bridge.

Remember how to specify graphs: Give vertex set, say when two vertices are adjacent.

Multigraphs--allow more than one edge between a pair of vertices. Only for this section.

```
In[18]:= GraphPlot[Join["a" -> # & /@ {"c", "c", "b", "b", "d"}, {"d" -> "c", "d" -> "b"}]]
```



A **trail** is a sequence of consecutively adjacent vertices using each edge at most once.
A **cycle** is a trail that ends where it begins.

Trails and cycles can repeat vertices, but not edges.

no repeated vertices	edges no repeated
path	trail
circuit	cycle

An **Euler cycle** is a cycle that uses all the edges and vertices.

A graph with an Euler cycle is **Eulerian**

How could a cycle use all the edges but not all the vertices? Degree 0.

There's a walk using all bridges exactly once \iff the graph is Eulerian.

- ❖ Why doesn't the Königsberg graph has an Euler cycle?

A multigraph has no Euler cycle if

- (a) it has an odd degree vertex

Or (b) it is not connected.

❖ **Theorem:** A multigraph G is Eulerian if and only if G is connected and has no odd degree vertex.

❖ If G has all even degrees, then the edges of G can be partitioned into circuits.

Proof: (by induction on the number of edges):

Base case: $e = 0$.

Induction step: $e > 0$.

Since all the degrees are even, by a previous result, G contains a circuit C.

Form G' by removing the edges of C.

The degree of any vertex on C has been reduced by two, so all vertices in G' still have even degree.

By induction the edges of G' can be partitioned into circuits.

Adding C to this partition gives a partition of the edges of G.

♣ **Theorem:** A multigraph G has an Euler cycle iff G is connected and every vertex has even degree.

Proof: \implies We can follow the cycle to travel between any two vertices, so G is connected.

If the Euler cycle passes through a vertex k times, then the vertex has degree $2k$, so all the degrees are even.

\impliedby If all degrees of G are even, then the edges of G can be partitioned into circuits.

Since the graph is connected these circuits can be joined together as a single cycle.

□ Is G Eulerian?

Yes

(a) all degrees are even and the graph is connected

OR (b) here's an Euler cycle.

No

(a) the graph is not connected

OR (b) here is a vertex of odd degree.

Application: Optimal route for recyclers.

Theorem guarantees there is an Euler cycle.

How to find it?--just do it. Patch together later.

Application: Optimal route for recyclers. Two overlapping squares+2 edges.

There's no Euler cycle--since odd degree vertices. But that doesn't mean there's no recycling.

Add an edge between the two odd degree vertices. Now the graph does have an Euler cycle.

Of course truck needs to follow roads rather than edge. Find the quickest route.

♣ **Theorem:** A connected multigraph G has an Euler trail iff G is connected and has exactly two vertices of odd degree.

Proof: \implies Passing through each vertex contributes two to its degree. Following the trail "passes through" all vertices except 1st and last.

\impliedby Let x and y be the two odd degree vertices. Form G' by adding (x,y) multiple edges allowed. Now find Euler cycle.

Remove (x,y) to obtain Euler trail.

- **Magic Trick**
- **Code**
- **DeBruijn Cycle**

Redo trick with 8 cards, asking 3 to stand.

Copy sequence to board, note that each of the 8 possible runs of 3 occurs exactly once.

I asked the reds from 5 people to stand. How many possible outcomes?

How many cards? Related.

I arranged the cards so that each possible R-B sequence of length 5 appeared exactly once.

Surprisingly cutting the cards doesn't affect the cyclic arrangement of the cards.

A binary sequence of length 2^n in which each possible subsequence of length n appears exactly once is called a DeBruijn cycle.

■ Constructing DeBruijn Cycles.

(Directed multigraphs.)

Find a binary sequence of length 8 where each subsequence of length 3 appears exactly once.

Create a vertex for each 2 digit sequence.

For each 3 digit sequence (i,j,k) add the directed edge $(ij) \rightarrow (jk)$.

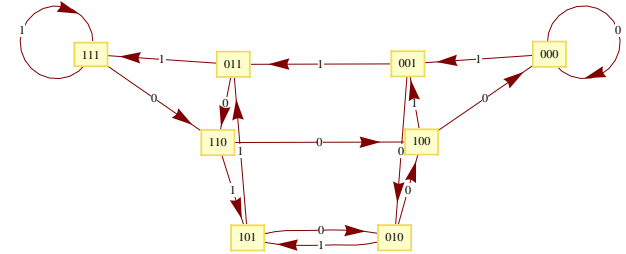
Label this edge with k .

Find a directed Euler cycle. The labels on the edges form a DeBruijn Cycle.

```
In[139]:= debriujnGraph[bits_] :=
Cases[Tuples[Range[2^bits-1] - 1, 2] /. x_Integer -> IntegerDigits[x, 2, bits - 1],
{{a_, b_}, {b_, c_}} :>
{{StringJoin@@ToString /@ {a, b} -> StringJoin@@ToString /@ {b, c}, c}}];

GraphPlot[debruijnGraph[4], DirectedEdges -> True]
```

Out[140]=



■ Industrial application.

Robotic forklift. Follows a line painted on the floor. Needs to know where it's located. Create a DeBruijn cycle with 2^{16} red/black dots spaced 1/4 inch apart. The robot reads a four inch strip of the dots directly below it. Based on these 16 dots it knows where it is to within 1/4 inch.

Read 16 dots 1/4 inch apart. Total of 4 inches.

```
totalDots = 2^16
totalInches = totalDots / 4
totalFeet = N[totalDots / 12]
totalMiles = totalFeet / 5280

65 536
16 384
5461.33
1.03434
```

Read 32 dots 1/4 inch apart. Total of 8 inches.

```

totalDots = 232
totalInches = totalDots / 4
totalFeet = N[totalDots / 12]
totalMiles = totalFeet / 5280

4 294 967 296

1 073 741 824

3.57914 × 108

67 786.7
    
```

2.2 Hamilton circuits.

2.2: 4(Find, with proof, two Hamilton graphs and two non-Hamilton graphs),

- A Hamilton circuit is a circuit that uses each vertex.

Recall a circuit does not repeat vertices.
So each vertex is used exactly once.

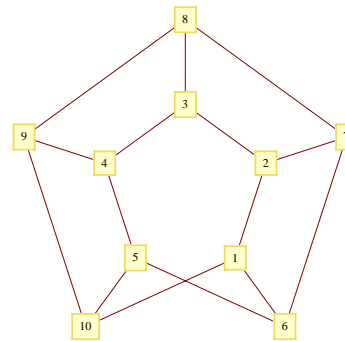
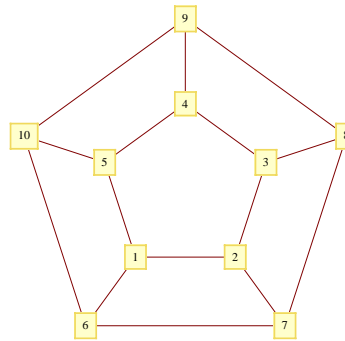
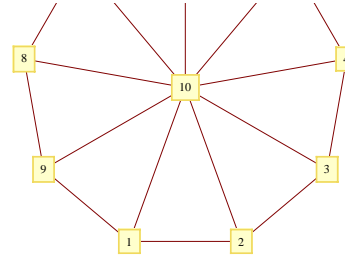
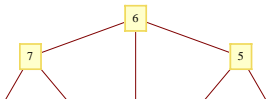
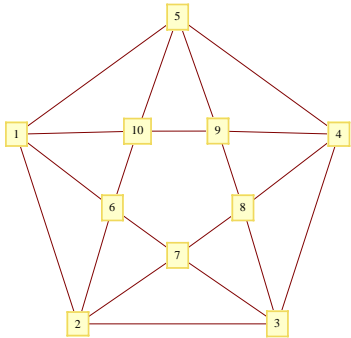
A graph with a Hamilton circuit is called "Hamilton" or "Hamiltonian".

- Example

```

In[78]:= GraphData /@ GraphData["Hamiltonian", 10][[{1, -1, -2, -3}]]
    
```

Out[78]= {



■ Is the graph Hamilton?

Yes options

(a) Give the circuit.

No options

(a) Careful analysis applying symmetry and rules.

■ Rules

If C is a Hamilton circuit in a graph G , then

Rule 1: If $\deg(v) = 2$, then both edges incident to v are in the circuit.

Rule 2: C has no proper subcircuits.

Rule 3: Any vertex v is incident to at most two edges on C .

Rule (1,3): Each vertex must have exactly two incident edges on the H-circuit.

```
In[96]:= nonHam[n_] := GraphData[Complement[GraphData["Connected", n], GraphData["Hamiltonian", n], GraphData["Tree", n]]
```

Redo example 2 on page 67.

Suppose G has a Hamilton circuit C .

By Rule 1, we use (b,a), (a,c), (e,g), (g,i) are on C .

By Rule 2, at most one of (b,d) and (d,c) is on C .

By symmetry (of original graph and our choices thus far), we may assume (b,d) and delete (d,c).

By Rule 1, (d,e) and (c,h) are on circuit.

By Rule 2, we delete (e,h).

By Rule 1 we use (

This violates Rule 1. Thus G has no Hamilton circuit.

■ Problem 3 on page 73.

First try 72 degree turn (f,k,h).

Then try 36 degree turn (f,k,g).

After all the work, just list the circuit.

■ Symmetry in figure 2.6: (Students do this at board.)

Delete n but remember that it is symmetrically attached to j,k,l,m .

Case 1) 180--Cuts work by 1/2

Case 2) 90 -- Cuts work by 1/4.

■ General Theorems

Yes options

(b) Theorem 1: If G is a connected graph on n vertices with all degrees at least $\frac{n}{2}$, then G is Hamilton.

Is $K_{3,4,7}$ Hamilton?

No options

(b) Theorem 3: Suppose G is planar. If G is Hamilton, then in any plane graph drawing

$$\sum (i-2)(r_i - r_i') = 0.$$

Draw a H-circuit, then surround with graph.

Compute (x_i') outside with degree i .

$$\text{Show } \sum (i-2)(x_i - x_i') = 0.$$

Show that a planar graph has no H-circuit, by showing that the sum can not be zero.

■ Proof of Grinberg's theorem

$$\text{Show } \sum (i-2)x_i = \sum (i-2)x_i'.$$

Draw example, add chord inside to persuade that the sum must be constant.

Prove that $\sum (i-2)x_i = n-2$. Leave $\sum (i-2)x_i'$ as exercise for topology.

For any circuit C with chords, let $I(C) = \sum (i-2)r_i$.

Prove $I(C) = n-2$ by induction on the number of chords.

■ Prove Theorem 4 : Every tournament has a H-path.

Prove Theorem 4 : Every tournament has a H-path.

(Definition of a tournament. Always know the definitions. This will be on the review.)

Proof by induction on n , the number of vertices.

Inductive step: Let G be a tournament on n vertices. Form G' by removing z .

G' is a tournament so it contains an H-path $H = (x_1, \dots, x_{n-1})$.

Case 1) $z \rightarrow x_1$ or $x_n \rightarrow z$.

Case 2) $x_1 \rightarrow z$ and $z \rightarrow x_n$.

Let k be the maximum such vertex so that $x_k \rightarrow z$

We know $1 \leq k \leq n-1$. Since k is maximum, we know $z \rightarrow x_{k+1}$.

G contains the H-path