Name______________________________ Due Date:_______________

(Q1) Now compute $A\begin{pmatrix}-2\\1\end{pmatrix}$ by moving $\boldsymbol{x}$ to $\begin{pmatrix}-2\\1\end{pmatrix}$. Answer: $A\begin{pmatrix}-2\\1\end{pmatrix} =$

(Q2) Where does A send the vector $\begin{pmatrix}3\\1\end{pmatrix}$?

(Q3) Use the graphic to find $\boldsymbol{x}$ that solves $A\,\boldsymbol{x} = \begin{pmatrix}-3\\2\end{pmatrix}$.

(Q4) Solve $A\,\boldsymbol{x} = \begin{pmatrix}-4\\3\end{pmatrix}$.

(Q5) Estimate the angle that $\boldsymbol{u}$ makes with the x-axis when $\theta = \frac{\pi}{2}$.

(Q6) What's the exact sum of these two angles? $\theta_1 + \theta_2 =$ ___.

(Q7) It turns out that for any matrix A, $\theta_1 + \theta_2 = \pi$. So for some A, if $\theta_1 < \frac{\pi}{2}$, then $\theta_2 >$ ___ and if $\theta_1 > \frac{\pi}{2}$, then $\theta_2 <$ ___. So either way there must be a $\boldsymbol{u}$ between $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ where $\theta =$ ___.

(Q8) The solution to $A\,\boldsymbol{x} = \boldsymbol{y}$ is $\boldsymbol{x} =$ ___$\boldsymbol{u}$ + ___$\boldsymbol{v}$.

(Q9) How much $A\,\boldsymbol{u}$ and $A\,\boldsymbol{v}$ add up to $\boldsymbol{y}$? Answer: $\boldsymbol{y} =$ __ $(A\,\boldsymbol{u})$ + __ $(A\,\boldsymbol{v})$.

(Q10) The solution to $A\,\boldsymbol{x} = \boldsymbol{y}$ is $\boldsymbol{x} =$ __ $\boldsymbol{u}$ + __ $\boldsymbol{v}$.

(Q11) The singular values of A are __ and __.

(Q12) Move $\boldsymbol{x}$ to $\boldsymbol{u}_1 + 2\,\boldsymbol{u}_2$ to verify $A(\boldsymbol{u}_1 + 2\,\boldsymbol{u}_2) =$ __ $\boldsymbol{v}_1$ + __ $\boldsymbol{v}_2$.

(Q13) The graphic shows

$A\,\boldsymbol{x} = A\,\boldsymbol{u}_1 + A\,\boldsymbol{u}_2 =$ __ $\boldsymbol{v}_1$ + __ $\boldsymbol{v}_2$.

(Q14) The singular values of A are __ and __.

(Q15) Use the singular values of A to compute $A(3\,\boldsymbol{u}_1 + 5\,\boldsymbol{u}_2) =$ __ $\boldsymbol{v}_1$ + __ $\boldsymbol{v}_2$.

(Q16) The singular values of A are __ and __.

(Q17) Step 1, find the $\boldsymbol{u}_i$ coords for $\boldsymbol{w}$: $\boldsymbol{w} =$ __ $\boldsymbol{u}_1$ + __ $\boldsymbol{u}_2$.

(Q18) Step 3, use these numbers as $\boldsymbol{v}$-coords for $A\,\boldsymbol{w}$: $A\,\boldsymbol{w} =$ __ $\boldsymbol{v}_1$ + __ $\boldsymbol{v}_2$.

(Q19) In terms of the vectors $\{\boldsymbol{z}_1, \boldsymbol{z}_2\}$ and $\{\boldsymbol{w}_1, \boldsymbol{w}_2\}$, the singular value decomposition for A is $A = (\quad | \quad)\begin{pmatrix} & \end{pmatrix}\begin{pmatrix} \text{——} \end{pmatrix}$.

(Q20) And $A\begin{pmatrix}\frac{-12}{13}\\ \frac{5}{13}\end{pmatrix} =$

(Q21) The graphic shows that the singular values are $s_1 =$ __ and $s_2 =$ __.

(Q22) Step 1: Find the $\boldsymbol{v}_i$ coordinates for $\boldsymbol{y}$: $\boldsymbol{y} =$ __ $\boldsymbol{v}_1$ + __ $\boldsymbol{v}_2$.

(Q23) Step 3: Use these numbers as $\boldsymbol{u}_i$ coordinates.
The solution to $A\,\boldsymbol{x} = \boldsymbol{y}$ is $\boldsymbol{x} =$ __ $\boldsymbol{u}_1$ + __ $\boldsymbol{u}_2$.

(Q24) $\boldsymbol{y} =$ ___ $\boldsymbol{v}_1$ + ___ $\boldsymbol{v}_2$

(Q25) $\boldsymbol{x} =$ ___ $\boldsymbol{u}_1$ + ___ $\boldsymbol{u}_2$

(Q26) The solution to $A\,\boldsymbol{x} = \begin{pmatrix}4\\2\end{pmatrix}$ is $\boldsymbol{x} = \begin{pmatrix} \\ \end{pmatrix}$.

(Q27) Where does matrix A sends all points on the blue line?

(Q28) Where does matrix A sends all points on the green line?

(Q29) From the picture you can see that $s_1 =$ ___ and $s_2 =$ ___ and $\boldsymbol{y} =$ ___$\boldsymbol{v}_1$.

(Q30) In terms of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, give the general solution to $A\,\boldsymbol{x} = \boldsymbol{y}$.
Answer: $\boldsymbol{x} =$ ___ $\boldsymbol{u}_1$ + ___ $\boldsymbol{u}_2$.

(Q31) The plot shows that the singular values are $s_1 =$ __ and $s_2 =$ __.

(Q32) The plot shows $\boldsymbol{y} =$ __ $\boldsymbol{v}_1$ + __ $\boldsymbol{v}_2$.

(Q33) The vector $\hat{\boldsymbol{y}}$ in the column space of A closest to $\boldsymbol{y}$ is $\hat{\boldsymbol{y}} =$ __ $\boldsymbol{v}_1$ + __ $\boldsymbol{v}_2$.

(Q34) How many solutions does $A\,\boldsymbol{x} = \boldsymbol{y}$ have? ___
How many solutions will $A\,\boldsymbol{x} = \hat{\boldsymbol{y}}$ have? ___

(Q35) The general solution to $A\,\boldsymbol{x} = \hat{\boldsymbol{y}}$ is $\boldsymbol{x} = \_\_\,\boldsymbol{u}_1 + \_\_\,\boldsymbol{u}_2$.

(Q36) The general least squares solution to $A\,\boldsymbol{x} = \boldsymbol{y}$ is $\boldsymbol{x} = \_\_\,\boldsymbol{u}_1 + \_\_\,\boldsymbol{u}_2$.

(Q37) $\boldsymbol{y} = \_\_\_\,\boldsymbol{v}_1 + \_\_\_\,\boldsymbol{v}_2$

(Q38) $\hat{\boldsymbol{y}} = \_\_\_\,\boldsymbol{v}_1 + \_\_\_\,\boldsymbol{v}_2$

(Q39) How many solutions will $A\,\boldsymbol{x} = \boldsymbol{y}$ have? ___
How many solutions will $A\,\boldsymbol{x} = \hat{\boldsymbol{y}}$ have? ___

(Q40) $\boldsymbol{x} = \_\_\_\,\boldsymbol{u}_1 + \_\_\_\,\boldsymbol{u}_2$

(Q41) Multiply out to get a numerical answer. There are NO solutions to $A\,\boldsymbol{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. But the least squares solutions are $\boldsymbol{x} =$

(Q42) Suppose $A = (\,\boldsymbol{v}_1 \mid \boldsymbol{v}_2 \mid \boldsymbol{v}_3\,) \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \hline \boldsymbol{u}_2 \\ \hline \boldsymbol{u}_3 \\ \hline \boldsymbol{u}_4 \end{pmatrix}$. In terms of the $\boldsymbol{u}_i$ and or $\boldsymbol{v}_i$:

a. Give a basis for the nullspace of A.
b. Give a basis for the column space of A.
c. Give the general least squares solution to $A\,\boldsymbol{x} = 10\,\boldsymbol{v}_1 + 9\,\boldsymbol{v}_2 + 8\,\boldsymbol{v}_3$.

(Q43) Suppose $A = (\,\boldsymbol{v}_1 \mid \boldsymbol{v}_2 \mid \boldsymbol{v}_3\,) \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \hline \boldsymbol{u}_2 \\ \hline \boldsymbol{u}_3 \end{pmatrix}$.
Express A^{-1} as a product of three matrices.

(Q44) Suppose $A = (\,\boldsymbol{v}_1 \mid \boldsymbol{v}_2 \mid \boldsymbol{v}_3\,) \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \hline \boldsymbol{u}_2 \\ \hline \boldsymbol{u}_3 \\ \hline \boldsymbol{u}_4 \end{pmatrix}$.

Express A^{+} as a product of three matrices.

(Q45) Consider the singular value decomposition

$$A = \left(\begin{array}{c|c|c|c} -\frac{5}{6} & -\frac{3}{6} & \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{6} & \frac{3}{6} & -\frac{1}{6} & \frac{5}{6} \\ -\frac{1}{6} & \frac{3}{6} & \frac{5}{6} & -\frac{1}{6} \\ \frac{3}{6} & -\frac{3}{6} & \frac{3}{6} & \frac{3}{6} \end{array}\right) \begin{pmatrix} \frac{2}{5} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left(\begin{array}{ccc} -\frac{14}{15} & \frac{2}{15} & \frac{5}{15} \\ \hline \frac{5}{15} & \frac{10}{15} & \frac{10}{15} \\ \hline \frac{2}{15} & -\frac{11}{15} & \frac{10}{15} \end{array}\right).$$

a. Give a basis for the nullspace of A.
b. Give a basis for the column space of A.
c. For the vector $\boldsymbol{y} = \begin{pmatrix} 0 \\ 18 \\ 18 \\ 0 \end{pmatrix}$, find $\hat{\boldsymbol{y}}$ the projection of $\boldsymbol{y}$ on the column space of A.
d. Express the pseudo-inverse A^{+} as a product of three matrices.
e. Give all least squares solutions to $A\,\boldsymbol{x} = \begin{pmatrix} 0 \\ 18 \\ 18 \\ 0 \end{pmatrix}$.

(Q46) Every matrix A has a singular value decomposition and to find it you usually have to use a machine. But if you like, this problem will show you how to find an SVD for $A = \begin{pmatrix} 54 & -28 \\ -25 & 0 \\ -6 & 42 \end{pmatrix}$ by hand. An SVD for A will have the form $A = (\,\boldsymbol{v}_1 \mid \boldsymbol{v}_2 \mid \boldsymbol{v}_3\,) \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \hline \boldsymbol{u}_2 \end{pmatrix}$.

a. Start with the vectors $\boldsymbol{p}_1 = \begin{pmatrix} t \\ 1-t \end{pmatrix}$ and $\boldsymbol{p}_2 = \begin{pmatrix} t-1 \\ t \end{pmatrix}$. For any value of t these vectors are orthogonal. Find a value of t so that $A\,\boldsymbol{p}_1$ and $A\,\boldsymbol{p}_2$ are orthogonal. I.e., compute $A\,\boldsymbol{p}_1$ and $A\,\boldsymbol{p}_2$, find their dot product, and then find a value of t that makes the dot product zero.
b. Once you've found t, you can get $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ by normalizing (making unit vectors out of) $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$.
c. You can get $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ by normalizing (making unit vectors out of) $A\boldsymbol{p}_1$ and $A\boldsymbol{p}_2$.
d. You can get s_1 by computing how much A stretched $\boldsymbol{p}_1$. I.e. compute $s_1 = \frac{\|A\,p_1\|}{\|p_1\|}$, similarly for s_2.
e. Use the cross product to get a vector $\boldsymbol{v}_3$ to complete the orthonormal basis of $\mathbb{R}^3$.