College of Science and Allied Health Undergraduate Research Project

A Modified Numerical Method

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Abstract

The adaptive stencil linear deviation method is introduced to find accurate approximations to solutions of partial differential equations (PDEs) known as wave equations. These PDE's describe how wave information propagates from one point in time to the next by modeling displacement from equililbrium. Initially, the exact position of the wave is known, and the position of the wave at future time points is to be determined. In practice, however, the exact position may only be known at a finite number of points in space.

One of the primary focuses of the research involves approximating the space derivative of the displacement variable. Finite differences can be used to approximate derivatives using a finite number of data points. In fact, when using four points in space to approximate the first derivative, there are three methods that could be used, each using different data sets. The way in which one selects from among these three derivative approximations can greatly affect the accuracy of the approximation (especially if the data exhibits shocks or discontinuities). We have developed a method for selecting the approximation based on the amount of linearity in the data. It has been shown that this Linear Deviation method outperforms standard methods.

Introduction The purpose of this research was to find accurate approximations to solutions of hyperbolic partial differential equations (PDEs). Hyperbolic PDEs, including wave equations, are used to model nearly all finite speed wave transmissions such as sound, radar, sonar, and light. One example of a hyperbolic PDE is the acoustic wave equation which models a three dimensional, multi-way wave. In [4], it is shown that the larger acoustic problem can be approximated by a collection of one-dimensional, one-way wave equations

$$u_t + u_x = 0 \qquad u(x,0) = f(x). \tag{1}$$

In this equation, $u(x,t)$ measures the displacement of the wave from rest at each point in space (x) and time (t). This equation describes how the wave changes from one point in time to the next. Initially ($t = 0$), we know the exact position of the wave at any point in time, so we are left to determine the position of the wave at future times ($t > 0$). One added complication is the fact that we are only given a finite amount of information (the value at a limited number of points).

Methods To approximate the value of the solution u at points in time after the initial time, we used a the Taylor Series expansion [1,3]

$$u(x, t + \Delta t) = u(x,t) + \Delta t u_t(x,t) + \frac{(\Delta t)^2}{2!} u_{tt}(x,t) + \ldots \tag{2}$$

We can approximate the value of a function at a later point in time provided we know the derivative values. Since, we do not have the entire function, (only a finite number of data points in time and space), we need to approximate these derivatives.

The initial data provides information in space, but at only a single time, making time derivative approximations difficult. From the PDE (1) we know $u_t = -u_x$, from which follows that $u_{tt} = u_{xx}$. Substituting this into the Taylor Series (2), we get the following expansion

$$u(x, t + \Delta t) = u(x,t) - \Delta t u_x(x,t) + \frac{(\Delta t)^2}{2!} u_{xx}(x,t) - \ldots \tag{3}$$

This new equation provides a method of determining the position of the wave at later points in time in terms of space derivatives. Therefore, one of the primary focuses of the research involved approximating the space derivative u_x. Finite differences [1,3] can be used to approximate derivatives using a finite number of data points. In fact, when using four points in space to approximate the u_x, there are three methods (stencils) that could be used to approximate u_x, each using different data sets. The way in which one selects from among these three derivative approximations is another of our primary interests.

One method (developed in the mid 1990s, see [4]) is referred to as the ENO method. ENO first selects from two sets of three points, selecting the set that forms the flatter interpolating parabola. It then selects one of two points to add to the set of three.

As a result of our research, we have found an alternative to the ENO method. This new method, the 2-pt method, selects the data with the least linear deviation. It does this by forming a line between two points and then selecting the two additional points (from a set of 4 points) which are closest to the line.

With either method (ENO or the 2-pt) we can approximate the second derivative of u with respect to x (u_{xx}) in two different ways. With each method (ENO or the 2-pt), one could first find u_x and then reapply the same method to the u_x data to arrive at an approximation to u_{xx}. A second approach would be to modify the routines so that the second derivative is approximated directly. The first approach requires nearly twice as many calculations. Acceptable increases in computational complexity are those increases which lead to a better approximation. As mentioned in the results below, the more computationally complex method does not lead to significant gains and is thus inferior.

An alternative to the Taylor series approach (2) is the Runge-Kutta [1,3] approach.

$$u(x, t + \Delta t) = u(x, t) + \Delta t(-b_1 u_x(x, t) - b_2(u_x - \Delta t a u_x)_x) \quad (4)$$

where a, b_1, and b_2 satisfy certain constraints. The results section explains why this approach did not lead to increased accuracy.

Results and Conclusions To find which method does better, we test the various methods against a set of problems for which we can determine the exact solution. We then compare the approximation to the exact using the l_1, l_2 and the l_∞ norms [1]. The l_1 norm sums up the differences between the true solution and the approximation. The l_2 norm calculates the sum of the squared differences between the true solution and the approximation. The l_∞ norm finds the maximum difference between the true solution and the approximation.

Using these norms to calculate error and the amount of work (number of calculations needed) to calculate u_x and u_{xx}, we were able to compare the various methods.

In this research, a series of modifications were made to the original ENO. One modification was using the first derivative, u_x, the second derivative, u_{xx}, and the third derivative, u_{xxx}, to approximate the solution. Another modification was in how the derivatives were approximated, whether the same methods were reapplied or the derivatives were calculated directly. The last modification was using either the Taylor series approach (3) or the Runge-Kutta approach (4)to approximate the derivatives.

In addition, we created our own (2-pt) method and analyzed the changes in accuracy/complexity when this method was subjected to the same modifications.

The results of these modifications to ENO and the 2-pt method are as follows. When using the first derivative, ENO did well, but the 2-pt method did poorly. Using the second derivative, ENO did not improve much while the 2-pt method improved greatly doing better than ENO. Using the third derivative, both methods improved somewhat if larger time steps were used, but had results similar to the second derivative if small time steps were taken. In all cases, calculating the derivatives directly did better than reapplying the methods. Also, the Runge-Kutta approach did not do as well as the Taylor Series approach.

In conclusion, we have found that there are significant gains when using the second derivative term in the Taylor Series time expansion when using the data to calculate both u_x and u_{xx} directly. In this setting, the 2-pt method does better than the ENO method with a comparable amount of calculations. We also found that the Runge-Kutta approach doesn't seem necessary since by using u_x and a linear operation, it collapses down to the Taylor Series time expansion.

References

[1] R.L. Burden and J.D. Faires. *Numerical Analysis.* Prindle, Weber & Schmidt, Boston, 3rd edition, 1985.

[2] J.C. Butcher. *The Numerical Analysis of Ordinary Differential Equations.* John Wiley & Sons Ltd., Great Britain, 1987.

[3] W. Chaney and D. Kincaid. *Numerical Mathematics and Computing.* Brooks/Cole, California, 3rd edition, numerical methods

[4] Robert Hoar. *An Adaptive Stencil Finite Difference Method for First Order Linear Hyperbolic Systems.* PhD thesis, Montana State University, 1995.

Other Presentations

I have given a presentation at the Argonne National Laboratory Symposium for Undergraduates in Science, Engineering, and Mathematics that took place on November 5 - 6, 1999. I have also giving this presentation to the Math Club here at the University of Wisconsin La Crosse.

Meeting Information

The Sixty-Eighth Annual Meeting of the MAA Wisconsin Section will take place at the University of Wisconsin - Superior on April 14 - 15, 2000. It is a state event for both undergraduates and professors of mathematics in which I will be giving an oral presentation on the research that I did last summer. Myself and many of the professors in the mathematics department along with some students will be driving to this event.

Budget

I will be staying at a hotel for two nights at \$52.00 per night. I will be traveling 500 miles at \$0.19/mile. This totals \$1'99.00 and the Mathematics Department has agreed to cover all other costs associated with this trip.

fund \$46 for gas.

\$150.