1. $\lim_{x\to 0} (1+x)^{\frac{1}{\sin x}}$

$= e^{\lim_{x\to 0} \ln (1+x)^{\frac{1}{\sin x}}}$

$= e^{\lim_{x\to 0} \frac{\ln(1+x)}{\sin x}}$

$\overset{L'H}{=} e^{\lim_{x\to 0} \frac{\frac{1}{1+x}}{\cos x}}$

$= e^1 = e$

2. $\lim_{x\to 0} \left(\frac{1+x}{\sin x} - \frac{1}{x} \right)$

$= \lim_{x\to 0} \frac{x+x^2-\sin x}{x \sin x}$

$\overset{L'H}{=} \lim_{x\to 0} \frac{1+2x-\cos x}{\sin x + x\cos x}$

$= \lim_{x\to 0} \frac{\frac{1-\cos x}{x} + 2}{\frac{\sin x}{x} + \cos x}$

$= \frac{\lim_{x\to 0} \frac{1-\cos x}{x} + 2}{1+1}$

$\overset{L'H}{=} \frac{\lim_{x\to 0} \frac{\sin x}{1} + 2}{2}$

$= 1$

3. $f(x) = (e^x-1)(e^{2x}-2)\cdots(e^{nx}-n)$

$f'(x) = e^x(e^{2x}-2)\cdots(e^{nx}-n) + (e^x-1)\cdot e^{2x}\cdot 2(e^{3x}-3)\cdots(e^{nx}-n) + \cdots$
$+ (e^x-1)\cdots(e^{(n-1)x}-(n-1))\cdot e^{nx}\cdot n$

$f'(0) = e^0(1-2)(1-3)(1-4)\cdots(1-n) + 0 + 0 + \cdots + 0$

$= (-1)(-2)(-3)\cdots(-(n-1))$

$= (-1)^{n-1}\cdot(n-1)!$

4. ① f is continuous at $x=1$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) \Rightarrow \boxed{b=0}$$

② $f'_-(1) = f'_+(1)$.

$$\cos[a(x-1)]\cdot a \Big|_{x=1} = \frac{1}{x}\Big|_{x=1}$$

$$\Rightarrow \boxed{a=1}$$

5. $$x^2-y+1 = e^y \Rightarrow 2x - y' + 0 = e^y \cdot y'$$

$$\Rightarrow y' + e^y y' = 2x$$

$$\Rightarrow y' = \frac{2x}{1+e^y}$$

6. Let $f(x) = \left(1+\frac{1}{x}\right)^x - e$ $\qquad \lim_{x\to 0^+} f(x) = 1-e < 0$

$$\lim_{x\to\infty} f(x) = e-e = 0.$$

$$f'(x) = \left(1+\frac{1}{x}\right)^x \cdot \left[x\ln\left(1+\frac{1}{x}\right)\right]'$$

$$= \left(1+\frac{1}{x}\right)^x \cdot \left[\ln\left(1+\frac{1}{x}\right) + x\cdot\frac{1}{1+\frac{1}{x}}\cdot(-x^{-2})\right]$$

$$= \underbrace{\left(1+\frac{1}{x}\right)^x}_{>0} \cdot \Big[\underbrace{\ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}}_{g(x)>0}\Big]$$

$$\lim_{x\to 0^+} g(x) = \infty > 0, \quad g'(x) = \frac{1}{x+1} - \frac{1}{x} + \frac{1}{(x+1)^2} = \frac{x^2+2x-(x+1)^2}{x(x+1)^2} = \frac{-1}{x(x+1)^2} < 0$$

$g(x)\downarrow$, $\lim_{x\to\infty} g(x) = 0. \Rightarrow g(x) > 0 \Rightarrow f'(x) > 0 \Rightarrow f(x)\uparrow \Rightarrow f(x) < 0.$

7. <u>Proof</u>: $f(0)+f(1)+f(2)=3$, $f(3)=1$.

<u>Case 1</u>: $f(0)=f(1)=f(2)=1$: By MVT, $\exists\, \xi \in (0,3)$, s.t. $f'(\xi)=0$

<u>Case 2</u>: Not all $f(0), f(1)$ and $f(2)$ are equal. Then there is at least one of them <1 and there is at least one of them >1. Since f is continuous on $[0,3]$, there is $a \in (0,2)$ s.t. $f(a)=1$. Note $f(3)=1$.

By MVT, $\exists\, \xi \in (a,3)$ s.t. $f'(\xi)=0$.

8. $\lim\limits_{n\to\infty} n\left(\frac{1}{1^2+n^2}+\frac{1}{2^2+n^2}+\cdots+\frac{1}{n^2+n^2}\right)$

$$= \lim_{n\to\infty} n \sum_{i=1}^{n} \frac{1}{i^2+n^2} = \lim_{n\to\infty} n \sum_{i=1}^{n} \frac{\frac{1}{n^2}}{1+\left(\frac{i}{n}\right)^2}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{1+\left(\frac{i}{n}\right)^2} \cdot \frac{1}{n} = \int_0^1 \frac{1}{1+x^2}\,dx = \arctan x \Big|_0^1$$

$$= \boxed{\frac{\pi}{4}}$$

9. (1) $\int \left(1-\frac{1}{x^2}\right)\sqrt{x\sqrt{x}}\,dx$

$$= \int \left(1-\frac{1}{x^2}\right)\sqrt{x^{\frac{3}{2}}}\,dx$$

$$= \int \left(1-\frac{1}{x^2}\right) x^{\frac{3}{4}}\,dx$$

$$= \int x^{\frac{3}{4}} - x^{-\frac{5}{4}}\,dx = \frac{4}{7}x^{\frac{7}{4}} + 4x^{-\frac{1}{4}} + C.$$

9. (2). $\int \frac{x}{1+x^4}\,dx \xlongequal[du=2x\,dx]{u=x^2} \int \frac{1}{1+u^2}\cdot\frac{1}{2}\,du$

$\frac{1}{2}du = x\,dx$

$= \frac{1}{2}\int \frac{1}{1+u^2}\,du$

$= \frac{1}{2}\arctan u + C$

$= \frac{1}{2}\arctan x^2 + C$.

10. $I = \int_0^2 x\sqrt{2x-x^2}\,dx$

$= \int_0^2 x\sqrt{x(2-x)}\,dx$

$\xlongequal[t=2-x]{x=2-t} \int_2^0 (2-t)\sqrt{t(2-t)}\,d(2-t)$

$= \int_0^2 (2-t)\sqrt{t(2-t)}\,dt$

① $I = \int_0^2 x\sqrt{2x-x^2}\,dx = \int_0^2 x\sqrt{x(2-x)}\,dx$

② $I = \int_0^2 (2-t)\sqrt{t(2-t)}\,dt = \int_0^2 (2-x)\sqrt{x(2-x)}\,dx$

①+②: $2I = \int_0^2 2\sqrt{x(2-x)}\,dx$

$\Rightarrow I = \int_0^2 \sqrt{x(2-x)}\,dx = \int_0^2 \sqrt{2x-x^2}\,dx$

$= \int_0^2 \sqrt{1-(x-1)^2}\,dx$

Let $u = x-1$

$I = \int_{-1}^{1} \sqrt{1-u^2}\,du$

$= \frac{1}{2}\cdot\pi\cdot 1^2 = \boxed{\frac{\pi}{2}}$.