Outline of Solutions

#1. $\lim\limits_{x\to 0} \dfrac{x-\arctan x}{x^k} = c. \Rightarrow k>0$

L'Hôpital's Rule $\longrightarrow \lim\limits_{x\to 0} \dfrac{1-\frac{1}{1+x^2}}{kx^{k-1}} = c \overset{L'H}{\Longrightarrow} \lim\limits_{x\to 0} \dfrac{\frac{2x}{(1+x^2)^2}}{k(k-1)x^{k-2}} = c$

$$\Rightarrow \lim_{x\to 0} \frac{\frac{2}{(1+x^2)^2}}{k(k-1)x^{k-3}} = c$$

$$\Rightarrow \boxed{k=3}, \text{ and } \frac{2}{3(3-1)} = c \Rightarrow \boxed{c=\frac{1}{3}}$$

#2. $\lim\limits_{x\to 0^+}\left(2-\dfrac{\ln(1+x)}{x}\right)^{\frac{1}{x}} = e^{\lim\limits_{x\to 0^+}\frac{1}{x}\ln\left(2-\frac{\ln(1+x)}{x}\right)} = e^{\lim\limits_{x\to 0^+}\frac{\ln\left(2-\frac{\ln(1+x)}{x}\right)}{x}}$

$$\overset{L'H}{=} e^{\lim\limits_{x\to 0^+}\frac{\frac{1}{2-\frac{\ln(1+x)}{x}}\cdot(-1)\cdot\frac{\frac{1}{1+x}x-\ln(1+x)}{x^2}}{1}} = e^{\lim\limits_{x\to 0^+}\frac{1}{2-\frac{\ln(1+x)}{x}}\cdot(-1)\cdot\lim\limits_{x\to 0^+}\frac{\frac{x}{1+x}-\ln(1+x)}{x^2}}$$

$$\overset{L'H}{=} e^{\frac{-1}{2-\lim\limits_{x\to 0^+}\frac{\frac{1}{1+x}}{1}}\cdot\lim\limits_{x\to 0^+}\frac{x-(1+x)\ln(1+x)}{x^2+x^3}}$$

$$\overset{L'H}{=} e^{\frac{-1}{1}\cdot\lim\limits_{x\to 0^+}\frac{1-\ln(1+x)-(1+x)\frac{1}{1+x}}{2x+3x^2}} = e^{-1\cdot\lim\limits_{x\to 0^+}\frac{-\ln(1+x)}{2x+3x^2}}$$

$$= e^{\lim\limits_{x\to 0^+}\frac{\ln(1+x)}{2x+3x^2}} \overset{L'H}{=} e^{\lim\limits_{x\to 0^+}\frac{\frac{1}{1+x}}{2+6x}} = e^{\frac{1}{2}} = \boxed{\sqrt{e}}$$

#3. $\lim\limits_{n\to\infty} n\left[f(\tfrac{1}{n})-1\right] = \lim\limits_{n\to\infty}\dfrac{f(\frac{1}{n})-1}{\frac{1}{n}} = \lim\limits_{x\to 0^+}\dfrac{f(x)-1}{x}$

By $f(x)-x=e^{x(1-f(x))} \Rightarrow f(0)=1$ i.e. $y(0)=1$ $\qquad = \lim\limits_{x\to 0^+}\dfrac{f(x)-f(0)}{x-0} = f'(0)$

Apply Implicit Differention to $y-x=e^{x(1-y)}$

$\Rightarrow y'-1 = e^{x(1-y)}\cdot\left[1-y+x(0-y')\right]$

$\Rightarrow y'-1 = e^{x(1-y)}(1-y) - xe^{x(1-y)}y'$

$\Rightarrow y' = \dfrac{e^{x(1-y)}(1-y)+1}{1+xe^{x(1-y)}}$

$\Rightarrow f'(0) = y'(0) = \dfrac{0+1}{1+0} = \boxed{1}$

#4. $\sin(xy)+\ln(y-x)=x$, $P(0,1)$

Differentiate: $\cos(xy)\cdot(y+xy')+\frac{1}{y-x}\cdot(y'-1)=1$

$\Rightarrow \cos(xy)\cdot y+x\cos(xy)\cdot y'+\frac{1}{y-x}y'-\frac{1}{y-x}=1$

$\Rightarrow y'=\dfrac{\frac{1}{y-x}+1-y\cos(xy)}{x\cos(xy)+\frac{1}{y-x}} \Rightarrow y'\Big|_{at\,(0,1)}=\dfrac{1+1-1}{0+1}=1$

$\Rightarrow$ Tangent line: $y-1=1(x-0) \Rightarrow \boxed{y=x+1}$

#5. $f(x+y)=f(x)+f(y)+x^2y+xy^2$, Given $\lim\limits_{x\to0}\frac{f(x)}{x}=1$

① let $x=y=0$: $f(0)=f(0)+f(0)+0+0 \Rightarrow f(0)=0$

② $f'(0)=\lim\limits_{x\to0}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to0}\dfrac{f(x)}{x}=1$

③ $f(x+y)-f(x)=f(y)+x^2y+xy^2$

$f'(x)=\lim\limits_{y\to0}\dfrac{f(x+y)-f(x)}{y}=\lim\limits_{y\to0}\dfrac{f(y)+x^2y+xy^2}{y}=\lim\limits_{y\to0}\dfrac{f(y)}{y}+\lim\limits_{y\to0}\dfrac{x^2y+xy^2}{y}$

$=1+\lim\limits_{y\to0}x^2+xy$

$=1+x^2 \Rightarrow \boxed{f'(x)=1+x^2}$

#6. $\left.\begin{array}{l}f(x)\text{ is odd}\\ \text{(a) } f''\text{ exists on }[-1,1]\end{array}\right\} \Rightarrow f(0)=0$, by $f(1)=1$.

f satisfies Lagrange's MVT: $\exists\,\xi\in(0,1)$ s.t. $f'(\xi)=\dfrac{f(1)-f(0)}{1-0}=\dfrac{1-0}{1-0}=1$.

(b) $f(x)$ is odd $\Rightarrow f'(x)$ is even. ($\underbrace{f(x)+f(-x)=0}_{\text{odd}} \Rightarrow f'(x)+f'(-x)(-1)=0 \Rightarrow \underbrace{f'(x)=f'(-x)}_{\text{even}}$)

Let $g(x)=e^x(f'(x)-1)$. By (a): $g(\xi)=e^{\xi}(f'(\xi)-1)=0$, then

$$g(-\xi) = e^{-\xi}\left(f'(-\xi)-1\right) \overset{f' \text{ even}}{=\!=} e^{-\xi}\left(f'(\xi)-1\right) = 0$$

g satisfies Rolle's MVT: $\exists\, \eta \in (-\xi, \xi) \subset (-1,1)$, s.t.

$$g'(\eta) = 0, \text{ i.e., } e^{\eta}\left(f'(\eta)-1\right) + e^{\eta}\cdot f''(\eta) = 0$$

$$\Rightarrow f'(\eta) - 1 + f''(\eta) = 0 \Rightarrow f''(\eta) + f'(\eta) = 1.$$

#7. $f(x) = x^{2x} = e^{2x\ln x}$, $(0,1]$

$$f'(x) = e^{2x\ln x}\left(2\ln x + 2x\cdot\tfrac{1}{x}\right) = e^{2x\ln x}\left(2\ln x + 2\right)$$

$\Rightarrow$ critical # in the domain: $x = e^{-1} = \frac{1}{e}$.

f': $-$ on $(0, \frac{1}{e})$, $+$ on $(\frac{1}{e}, 1)$

f: $\searrow$ $\nearrow$

$$\Rightarrow \min(f) = f\left(\tfrac{1}{e}\right) = e^{2\cdot\frac{1}{e}\cdot\ln\frac{1}{e}} = \boxed{e^{-\frac{2}{e}}}$$

#8. $$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty}\frac{\sqrt[n]{n!}}{\sqrt[n]{n^n}} = \lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}} = \lim_{n\to\infty}\sqrt[n]{\frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdots\frac{n}{n}}$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdots\frac{n}{n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} e^{\ln\left(\frac{1}{n}\cdot\frac{2}{n}\cdots\frac{n}{n}\right)^{\frac{1}{n}}}$$

$$= e^{\lim\limits_{n\to\infty}\frac{1}{n}\ln\left(\frac{1}{n}\cdot\frac{2}{n}\cdots\frac{n}{n}\right)} = e^{\lim\limits_{n\to\infty}\frac{1}{n}\left(\ln\frac{1}{n} + \ln\frac{2}{n} + \cdots + \ln\frac{n}{n}\right)}$$

$$= e^{\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}\ln\frac{i}{n}} = e^{\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\ln\frac{i}{n}\cdot\frac{1}{n}} = e^{\int_0^1 \ln x\,dx} = \boxed{e^{-1}}$$

Note ~~[illegible]~~ $\underbrace{\int_0^1 \ln x\,dx}_{\text{improper integral}} = -1$ by $\lim\limits_{t\to 0^+}\int_t^1 \ln x\,dx$ and integration by parts.

#9. $\int_{-\infty}^{\infty} \underset{\substack{\downarrow \\ \text{even}}}{e^{k|x|}} dx = 2\int_0^{\infty} e^{kx} dx = 2 \cdot e^{kx} \cdot \frac{1}{k} \Big|_0^{\infty} = 1$

$\Rightarrow \lim_{x\to\infty} \frac{2}{k}e^{kx} - \frac{2}{k} = 1$

$\Rightarrow k<0$ and $-\frac{2}{k}=1 \Rightarrow k=-2$

#10. $\int_0^{\pi} \sqrt{x}\cos\sqrt{x}\, dx \xrightarrow[x=t^2]{\sqrt{x}=t} \int_0^{\sqrt{\pi}} t\cos t \cdot 2t\, dt$

$= \int_0^{\sqrt{\pi}} 2t^2 \cos t\, dt$ integration by parts.

$= 2t^2 \sin t \Big|_0^{\sqrt{\pi}} - \int_0^{\sqrt{\pi}} \sin t \cdot 4t\, dt$

$= 2\pi \sin\sqrt{\pi} - 4t(-\cos t)\Big|_0^{\sqrt{\pi}} \cancel{+} - \int_0^{\sqrt{\pi}} \cos t \cdot 4\, dt$

$= 2\pi\sin\sqrt{\pi} + 4\sqrt{\pi}\cos\sqrt{\pi} - 4\sin t\Big|_0^{\sqrt{\pi}}$

$= 2\pi\sin\sqrt{\pi} + 4\sqrt{\pi}\cos\sqrt{\pi} - 4\sin\sqrt{\pi}$

#11. $\int_1^{\infty} \frac{\ln x}{(1+x)^2} dx = \lim_{a\to\infty} \int_1^a \frac{\ln x}{(1+x)^2} dx$

integration by parts

$\int_1^a \frac{\ln x}{(1+x)^2} dx = \int_1^a -\ln x\, d\left(\frac{1}{1+x}\right) = -\ln x \cdot \frac{1}{1+x}\Big|_1^a + \int_1^a \frac{1}{1+x}\cdot\frac{1}{x} dx$

$= \frac{-\ln a}{1+a} + \int_1^a \frac{1}{x} - \frac{1}{1+x}\, dx$

$= \frac{-\ln a}{1+a} + [\ln x - \ln(1+x)]\Big|_1^a = \frac{-\ln a}{1+a} + \ln\frac{x}{1+x}\Big|_1^a$

$= \frac{-\ln a}{1+a} + \ln\frac{a}{1+a} - \ln\frac{1}{2}$

then $\lim_{a\to\infty} \frac{-\ln a}{1+a} + \ln\frac{a}{1+a} - \ln\frac{1}{2} = 0 + 0 - \ln\frac{1}{2} = \ln 2$

$\Rightarrow \boxed{\int_1^{\infty} \frac{\ln x}{(1+x)^2} dx = \ln 2}$