Key — 2014

1. $$\lim_{x\to 0}\left[\frac{1}{\ln(1+x)}-\frac{1}{x}\right] = \lim_{x\to 0}\frac{x-\ln(1+x)}{x\ln(1+x)} \overset{L'H}{=} \lim_{x\to 0}\frac{1-\frac{1}{1+x}}{\ln(1+x)+\frac{x}{1+x}}$$

$$= \lim_{x\to 0}\frac{1+x-1}{(1+x)\ln(1+x)+x} = \lim_{x\to 0}\frac{x}{(1+x)\ln(1+x)+x} \overset{L'H}{=} \lim_{x\to 0}\frac{1}{\ln(1+x)+1+1} = \frac{1}{2}$$

2. $$\lim_{x\to 0^+}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = e^{\lim_{x\to 0^+}\ln\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}} = e^{\lim_{x\to 0^+}\frac{\ln\frac{\sin x}{x}}{x^2}} = e^{\lim_{x\to 0^+}\frac{\ln\sin x-\ln x}{x^2}}$$

$$\overset{L'H}{=} e^{\lim_{x\to 0^+}\frac{\frac{1}{\sin x}\cos x-\frac{1}{x}}{2x}} = e^{\lim_{x\to 0^+}\frac{\frac{x\cos x-\sin x}{x\sin x}}{2x}} = e^{\lim_{x\to 0^+}\frac{x\cos x-\sin x}{2x^2\sin x}}$$

$$\overset{L'H}{=} e^{\lim_{x\to 0^+}\frac{\cos x-x\sin x-\cos x}{4x\sin x+2x^2\cos x}} = e^{\lim_{x\to 0^+}\frac{-x\sin x}{4x\sin x+2x^2\cos x}} = e^{\lim_{x\to 0^+}\frac{-1}{4+2\left(\frac{x}{\sin x}\right)\cos x}} = \boxed{e^{\frac{-1}{6}}}$$

3. $$f'_+(0) = \lim_{x\to 0^+}\frac{f(x)-f(0)}{x} = \lim_{x\to 0^+}\frac{\ln(x+e)-1}{x} \overset{L'H}{=} \frac{1}{e}$$

$$f'_-(0) = \lim_{x\to 0^-}\frac{f(x)-f(0)}{x} = \lim_{x\to 0^-}\frac{a^x-1}{x} \overset{L'H}{=} \ln a$$

$$\Rightarrow \ln a = \frac{1}{e} \Rightarrow a = e^{\frac{1}{e}} = \boxed{\sqrt[e]{e}}$$

4. $$\lim_{x\to 0}\frac{f(a-3x)-f(a)}{2x} = \lim_{x\to 0}\frac{f(a-3x)-f(a)}{-3x}\cdot\frac{1}{-\frac{2}{3}} \xrightarrow[x\to 0\ \Rightarrow h\to 0]{\text{Let } h=-3x} \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\cdot\left(-\frac{3}{2}\right)$$

$$\overset{\text{by definition}}{=} f'(a)\cdot\left(-\frac{3}{2}\right) = 1\cdot\left(-\frac{3}{2}\right) = \boxed{-\frac{3}{2}}$$

5. $f'(x) = 2x-2 \Rightarrow f(x) = x^2-2x+C,\ x\in[0,2] \Rightarrow f(2) = C$.

Note: f has period 4 $\Rightarrow f(2) = f(2-4) = f(-2) = C$
f is odd $\Rightarrow f(2) = -f(-2)$
$\Rightarrow C = 0$.

$\Rightarrow f(x) = x^2-2x,\ x\in[0,2]$.

then $f(7) = f(3+4) = f(3) = f(-1+4) = f(-1) \overset{\text{odd}}{=} -f(1) = -(1-2) = \boxed{1}$

6. $f(xy) = f(x) + f(y) \Rightarrow f'(xy)(y + xy') = f'(x) + f'(y) \cdot y'$ implicit differentiation

By $f(0) = 0$, $\quad f'(0) \cdot 0 = f'(0) + f'(0) \cdot f'(0)$

$\Rightarrow 0 = f'(0)(1 + f'(0)) \Rightarrow f'(0) = 0$ or $f'(0) = -1$

but $f'(x) \neq 0$, so $\boxed{f'(0) = -1}$.

7. (a) $\lim_{x \to \infty} f(x) = 2 \Rightarrow \forall \varepsilon > 0, \exists N$, s.t. $x > N$ and $|f(x) - 2| < \varepsilon$

$\Rightarrow f(N+1) > 1$

f is differentiable on $[0, \infty) \Rightarrow f$ is continuous on $[0, \infty)$. Then by the Intermediate Value Theorem, $\exists a \in (0, N+1)$ s.t. $f(a) = 1$.

(b) $f(0) = 0$, $f(a) = 1$ By Mean Value Theorem, $\exists \xi \in (0, a)$ s.t.

$$f'(\xi) = \frac{f(a) - f(0)}{a - 0} = \frac{1}{a}.$$

8. (a) $0 \le g(x) \le 1 \Rightarrow \int_a^x 0 \, dt \le \int_a^x g(t) \, dt \le \int_a^x 1 \, dt$

$\Rightarrow 0 \le \int_a^x g(t) \, dt \le x - a, \quad x \in [a, b]$

(b) Define $F(s) = \int_a^{a + \int_a^s g(t) dt} f(x) \, dx - \int_a^s f(x) g(x) \, dx$

To show $F(b) \le 0$, check $F(a) = 0$, it suffices to show $F(s)$ is decreasing, i.e., $F'(s) \le 0$, $s \in [a, b]$

$$F'(s) = f\left(a + \int_a^s g(t) dt\right) \cdot g(s) - f(s) g(s)$$

$$= g(s) \left[f\left(a + \int_a^s g(t) dt\right) - f(s) \right] \le 0$$

Because $a \le a + \int_a^s g(t) \, dt \le a + s - a = s$ by (a) and f is increasing, $1 \ge g(s) \ge 0$.

9. $\int \frac{\sin x \cos x}{1+\sin^4 x}\,dx \;\frac{u=\sin^2 x}{du=2\sin x\cos x\,dx}\; \int \frac{\frac{1}{2}\,du}{1+u^2}$

$= \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\arctan u + C$

$= \frac{1}{2}\arctan(\sin^2 x) + C$

10. $\int_{-\infty}^{1} \frac{1}{x^2+2x+5}\,dx = \lim_{t\to-\infty} \int_t^1 \frac{1}{x^2+2x+5}\,dx$

$= \lim_{t\to-\infty} \int_t^1 \frac{1}{(x+1)^2+4}\,dx = \lim_{t\to-\infty} \frac{1}{4}\int_t^1 \frac{1}{1+\left(\frac{x+1}{2}\right)^2}\,dx$

$= \lim_{t\to-\infty} \frac{1}{4}\cdot 2 \int_t^1 \frac{1}{1+\left(\frac{x+1}{2}\right)^2}\,d\left(\frac{x+1}{2}\right)$

$= \lim_{t\to-\infty} \frac{1}{2}\cdot \arctan\left(\frac{x+1}{2}\right)\Big|_t^1$

$= \lim_{t\to-\infty} \frac{1}{2}\left(\arctan 1 - \arctan\frac{t+1}{2}\right)$

$= \frac{1}{2}\left[\arctan 1 - \left(-\frac{\pi}{2}\right)\right]$

$= \frac{1}{2}\left(\frac{\pi}{4}+\frac{\pi}{2}\right) = \frac{1}{2}\cdot\frac{3\pi}{4} = \frac{3\pi}{8}$