#1. $\lim_{x\to 0} (\cos 2x + 2x\sin x)^{\frac{1}{x^2}}$

$$= \lim_{x\to 0} e^{\frac{1}{x^2}\ln(\cos 2x + 2x\sin x)}$$

$$= e^{\lim_{x\to 0} \frac{\ln(\cos 2x + 2x\sin x)}{x^2}}$$

$$\overset{L'H}{=} e^{\lim_{x\to 0} \frac{\frac{1}{\cos 2x + 2x\sin x}\cdot[-\sin 2x\cdot 2 + 2\sin x + 2x\cos x]}{2x}}$$

$$= e^{\lim_{x\to 0} \frac{1}{\cos 2x + 2x\sin x}\cdot\left[-\frac{\sin 2x}{x} + \frac{\sin x}{x} + \cos x\right]}$$

$$= e^{\lim_{x\to 0} \frac{1}{\cos 2x + 2x\sin x}\cdot\left[-\frac{\sin 2x}{2x}\cdot 2 + \frac{\sin x}{x} + \cos x\right]}$$

$$= e^{\frac{1}{1+0}\cdot[-1\cdot 2 + 1 + 1]}$$

$$= e^0$$

$$= \boxed{1}$$

Note: $\lim_{x\to 0} \frac{\sin x}{x} = 1$

#2. $$\lim_{x\to 0} \frac{\int_0^x t\ln(1+t\sin t)\,dt}{1-\cos x^2}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{x\ln(1+x\sin x)}{0+\sin x^2\cdot 2x}$$

$$= \lim_{x\to 0} \frac{\ln(1+x\sin x)}{2\sin x^2}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{\frac{1}{1+x\sin x}\cdot[\sin x + x\cos x]}{2\cos x^2\cdot 2x}$$

$$= \lim_{x\to 0} \frac{1}{1+x\sin x}\cdot\frac{1}{4\cos x^2}\cdot\left[\frac{\sin x}{x}+\cos x\right]$$

$$= \frac{1}{1+0}\cdot\frac{1}{4\cdot 1}\cdot[1+1]$$

$$= \boxed{\frac{1}{2}}$$

#3. $f(x) = \arctan x - \dfrac{x}{1+ax^2}$

$$\Rightarrow f'(x) = \frac{1}{1+x^2} - \frac{1-ax^2}{(1+ax^2)^2} \quad \text{by quotient Rule}$$

$$\Rightarrow f''(x) = \frac{-2x}{(1+x^2)^2} - \frac{-6ax+2a^2x^3}{(1+ax^2)^3} \quad \text{by Quotient Rule}$$

$$\Rightarrow f'''(x) = \frac{-2(1+x^2)^2 + 2x\cdot 2(1+x^2)\cdot 2x}{(1+x^2)^4} - \frac{(-6a+6a^2x^2)(1+ax^2)^3 - (-6ax+2a^2x^3)\,3\cdot(1+ax^2)^2\,2ax}{(1+ax^2)^6}$$

$$= \frac{-2(1+x^2)+8x^2}{(1+x^2)^3} - \frac{(-6a+6a^2x^2)(1+ax^2) - (-6ax+2a^2x^3)\,6ax}{(1+ax^2)^4}$$

$$\Rightarrow f'''(0) = -2+6a = 1 \Rightarrow 6a = 3$$

$$\Rightarrow \boxed{a = \frac{1}{2}}$$

#4 $f(x)=\begin{cases} x, & x\le 0 \\ \frac{1}{n}, & \frac{1}{n+1}<x\le\frac{1}{n} \end{cases}$ $\longrightarrow$ $x\le\frac{1}{n}<\frac{x}{1-x}$ $\Downarrow$ Squeeze Thm.

Note: $f(0)=0$, $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} x=0$, $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{1}{n}=0$

$$\Rightarrow \lim_{x\to 0} f(x)=f(0)=0 \Rightarrow f \text{ is continuous at } 0.$$

$$f'_-(0)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{x-0}{x-0}=1$$

$$f'_+(0)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{\frac{1}{n}-0}{x-0}=\lim_{x\to 0^+}\frac{\frac{1}{n}}{x}$$

By $x\le\frac{1}{n}<\frac{x}{1-x}$, $\Rightarrow 1\le\frac{\frac{1}{n}}{x}<\frac{1}{1-x}$. Note $\lim_{x\to 0^+}\frac{1}{1-x}=1$.

$\Rightarrow f'_+(0)=\lim_{x\to 0^+}\frac{\frac{1}{n}}{x}=1$ by Squeeze Theorem.

Above all, $f'_-(0)=f'_+(0)=1 \Rightarrow \boxed{f'(0)=1}$

#5. $f(x) = (x+1)^2 + 2\int_0^x f(t)\,dt$. Note: $f(0) = 1$

$\Rightarrow f'(x) = 2(x+1) + 2f(x)$

$\Rightarrow f''(x) = 2 + 2f'(x)$

$\Rightarrow f'''(x) = 2f''(x)$

$\Rightarrow f^{(4)}(x) = 2f'''(x)$

$\vdots$

$f^{(n)}(x) = 2f^{(n-1)}(x)$

$\Rightarrow f^{(n)}(x) = \overbrace{2\cdot 2\cdot 2\cdots 2}^{n-2} f''(x)$ if $n \geq 3$

But $f''(x) = 2 + 2f'(x) = 2 + 2[2(x+1) + 2f(x)] = 2 + 4(x+1) + 4f(x)$

$\Rightarrow f^{(n)}(x) = 2^{n-2} \cdot [2 + 4(x+1) + 4f(x)]$ if $n \geq 3$

$\Rightarrow f^{(n)}(0) = 2^{n-2}[2 + 4 + 4f(0)] = 2^{n-2} \cdot 10 = 2^{n-1} \cdot 5$ if $n \geq 3$.

If $n = 2$: $f''(0) = 2 + 4(0+1) + 4f(0) = 2 + 4 + 4 = 10 = 2^{2-1} \cdot 5$

$\Rightarrow f^{(n)}(0) = \underline{\underline{2^{n-1} \cdot 5}}$ for $n \geq 2$

#6 $\lim_{n\to\infty} \frac{1}{n^2}\left(\sin\frac{1}{n} + 2\sin\frac{2}{n} + \cdots + n\sin\frac{n}{n}\right)$

$$= \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} i \sin\frac{i}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n} \sin\frac{i}{n} \cdot \frac{1}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x \quad , \quad \Delta x = \frac{1-0}{n}, \quad x_i = 0 + i \cdot \Delta x$$

$$= \int_0^1 x \sin x \, dx$$

$$= \int_0^1 x \, d(-\cos x)$$

$$= x(-\cos x)\Big|_0^1 - \int_0^1 -\cos x \, dx$$

$$= -\cos 1 + \int_0^1 \cos x \, dx$$

$$= -\cos 1 + \sin x \Big|_0^1$$

$$= \boxed{\sin 1 - \cos 1}$$

#7 $f(x) = \int_0^1 |t^2 - x^2| \, dt \quad , \quad x > 0.$

Case I : $x \geq 1$.

then $f(x) = \int_0^1 x^2 - t^2 \, dt = (x^2 t - \frac{1}{3}t^3)\Big|_0^1 = x^2 - \frac{1}{3}$

$\Rightarrow f'(x) = 2x$

Case II : $0 < x < 1$

then $f(x) = \int_0^x |t^2 - x^2| \, dt + \int_x^1 |t^2 - x^2| \, dt$

$= \int_0^x x^2 - t^2 \, dt + \int_x^1 t^2 - x^2 \, dt$

$= (x^2 t - \frac{1}{3}t^3)\Big|_0^x + (\frac{1}{3}t^3 - x^2 t)\Big|_x^1$

$= x^3 - \frac{1}{3}x^3 + (\frac{1}{3} - x^2) - (\frac{1}{3}x^3 - x^3)$

$= \frac{2}{3}x^3 + \frac{1}{3} - x^2 + \frac{2}{3}x^3 = \frac{4}{3}x^3 - x^2 + \frac{1}{3}$

$\Rightarrow f'(x) = 4x^2 - 2x$

So $f'(x) = \begin{cases} 4x^2 - 2x, & 0 < x < 1 \\ 2x, & x \geq 1. \end{cases}$ $\Rightarrow$ critical numbers $\frac{1}{2}$, i.e. $f'(x) = 0$.

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then f': $0 \quad - \quad \frac{1}{2} \quad +$

f: $\searrow \quad \nearrow$

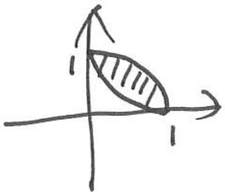
So $f(x)$ gets the minimum $f(\frac{1}{2}) =$

$= \int_0^1 |t^2 - \frac{1}{4}| \, dt$

$= \int_0^{\frac{1}{2}} \frac{1}{4} - t^2 \, dt + \int_{\frac{1}{2}}^1 t^2 - \frac{1}{4} \, dt$

$= (\frac{1}{4}t - \frac{1}{3}t^3)\Big|_0^{\frac{1}{2}} + (\frac{1}{3}t^3 - \frac{1}{4}t)\Big|_{\frac{1}{2}}^1 = \boxed{\frac{1}{4}}$

#8. ① $y=\sqrt{1-x^2}$ ② $x=\cos^3 t,\ y=\sin^3 t \Rightarrow y=\sqrt{(1-x^{\frac{2}{3}})^3}$



$$V=\int_0^1 \pi R^2-\pi r^2\,dx=\int_0^1 \pi(\sqrt{1-x^2})^2-\pi\left(\sqrt{(1-x^{\frac{2}{3}})^3}\right)^2 dx$$

$$=\int_0^1 \pi(1-x^2)-\pi(1-x^{\frac{2}{3}})^3\,dx$$

$$=\int_0^1 \pi(1-x^2)\,dx-\int_0^1 \pi(1-x^{\frac{2}{3}})^3\,dx$$

$$=\pi\left(x-\tfrac{1}{3}x^3\right)\Big|_0^1-\pi\int_0^1 (1-x^{\frac{2}{3}})^3\,dx$$

$$=\frac{2\pi}{3}-\pi\int_0^1 (1-x^{\frac{2}{3}})^3\,dx$$

$$\underset{dx=3\cos^2 t(-\sin t)dt}{\overset{x=\cos^3 t}{=\!=}}\ \frac{2}{3}\pi-\pi\int_{\frac{\pi}{2}}^{0}(1-\cos^2 t)^3\cdot 3\cos^2 t\,(-\sin t)\,dt$$

$$\overset{u=\cos t}{=\!=}\ \frac{2\pi}{3}-\pi\int_0^1 (1-u^2)^3\cdot 3u^2\,du$$

$$=\frac{2\pi}{3}-\pi\int_0^1 (1-u^6-3u^2+3u^4)\cdot 3u^2\,du$$

$$=\frac{2\pi}{3}-\pi\int_0^1 3u^2-3u^8-9u^4+9u^6\,du$$

$$=\frac{2\pi}{3}-\pi\cdot\left(u^3-\tfrac{1}{3}u^9-\tfrac{9}{5}u^5+\tfrac{9}{7}u^7\right)\Big|_0^1$$

$$=\frac{2\pi}{3}-\pi\left(1-\frac{1}{3}-\frac{9}{5}+\frac{9}{7}\right)$$

$$=\frac{2\pi}{3}-\frac{2}{3}\pi+\frac{9\pi}{5}-\frac{9\pi}{7}$$

$$=\left(\frac{9}{5}-\frac{9}{7}\right)\pi=\left(\frac{1}{5}-\frac{1}{7}\right)9\pi=\frac{2}{35}9\pi=\boxed{\frac{18\pi}{35}}$$

#9. $\displaystyle\int_1^9 \frac{(2\sqrt{x}+1)\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$

$u = 1+\sqrt{x}$

$du = \frac{1}{2}\frac{1}{\sqrt{x}}dx$

$2du = \frac{1}{\sqrt{x}}dx$

$$\int_2^4 \left[2(u-1)+1\right]\sqrt{u}\cdot 2du$$

$$= \int_2^4 (2u-1)\,2\sqrt{u}\,du$$

$$= \int_2^4 4u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\,du$$

$$= \left(\frac{8}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}}\right)\Big|_2^4$$

$$= \left(\frac{8}{5}\cdot 32 - \frac{4}{3}\cdot 8\right) - \left(\frac{8}{5}\cdot 4\sqrt{2} - \frac{4}{3}\cdot 2\cdot\sqrt{2}\right)$$

$$= \boxed{\frac{512}{15} - \frac{56\sqrt{2}}{15}}$$

#10. $\int_0^{\pi^{2/3}} \sqrt{x}\,\sin^2(x^{3/2})\cos^3(x^{3/2})\,dx$

$u = \sin(x^{3/2})$

$du = \cos(x^{3/2}) \cdot \frac{3}{2}\sqrt{x}\,dx$

$\frac{2}{3}du = \sqrt{x}\cos(x^{\frac{3}{2}})dx$

$\int_0^0 u^2(1-u^2)\cdot\frac{2}{3}\,du$, Note: $\sin 0 = 0$
$\sin\pi = 0$

$= 0$