#1 $\displaystyle \lim_{x\to 0^+} \frac{\int_0^x \sqrt{x-t}\, e^t\, dt}{\sqrt{x^3}}$

$$\overset{u=x-t}{=} \lim_{x\to 0^+} \frac{\int_x^0 \sqrt{u}\, e^{x-u}\, d(x-u)}{\sqrt{x^3}}$$

$$= \lim_{x\to 0^+} \frac{\int_x^0 \sqrt{u}\cdot e^x\cdot e^{-u}\, d(-u)}{\sqrt{x^3}}$$

$$= \lim_{x\to 0^+} \frac{e^x\cdot \int_0^x \sqrt{u}\cdot e^{-u}\, du}{\sqrt{x^3}}$$

$$\overset{L'H}{=} \lim_{x\to 0^+} \frac{e^x\cdot \int_0^x \sqrt{u}\cdot e^{-u}\, du + e^x\cdot \sqrt{x}\cdot e^{-x}}{\frac{3}{2}x^{\frac{1}{2}}}$$

$$= \lim_{x\to 0^+} \frac{e^x\cdot \int_0^x \sqrt{u}\cdot e^{-u}\, du}{\frac{3}{2}x^{\frac{1}{2}}} + \lim_{x\to 0^+} \frac{e^x\cdot e^{-x}}{\frac{3}{2}}$$

$$= \lim_{x\to 0^+} \frac{e^x\cdot \int_0^x \sqrt{u}\, e^{-u}\, du}{\frac{3}{2}x^{\frac{1}{2}}} + \frac{1}{\frac{3}{2}}$$

$$\overset{L'H}{=} \lim_{x\to 0^+} \frac{e^x\cdot \int_0^x \sqrt{u}\, e^{-u}\, du + e^x\cdot \sqrt{x}\, e^{-x}}{\frac{3}{2}\cdot\frac{1}{2}x^{-\frac{1}{2}}} + \frac{2}{3}$$

$$= \lim_{x\to 0^+} \frac{x^{\frac{1}{2}} e^x\cdot \int_0^x \sqrt{u}\, e^{-u}\, du + e^x\cdot x\cdot e^{-x}}{\frac{3}{4}} + \frac{2}{3}$$

$$= 0 + \frac{2}{3}$$

$$= \boxed{\frac{2}{3}}$$

#2. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} b = b$, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1-\cos\sqrt{x}}{ax}$

$$\stackrel{L'H}{=} \lim_{x\to 0^+} \frac{0+\sin\sqrt{x}\cdot\frac{1}{2}x^{-\frac{1}{2}}}{a} = \lim_{x\to 0^+} \frac{\sin\sqrt{x}}{2a\sqrt{x}} = \frac{1}{2a}, \text{ by } \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

$f(x)$ is continuous at $0 \Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$$\Rightarrow b = \frac{1}{2a} \Rightarrow 2ab = 1 \Rightarrow \boxed{ab = \frac{1}{2}}.$$

#3. (a) Proof: $\lim_{x\to 0^+} \frac{f(x)}{x} < 0 \Rightarrow \lim_{x\to 0^+} f(x) = 0$ and $f'(x) < 0$ as $x \to 0^+$

$\Rightarrow$ there exists $a > 0$ and $f(a) < 0$. Given $f(1) > 0$, by Intermediate Value Theorem, $f(x) = 0$ has at least one solution in $(a,1)$, i.e., in $(0,1)$.

(b) By (a), assume $f(b) = 0$ and $b \in (0,1)$. Let $g(x) = f(x)\cdot f'(x)$.

then $g(0) = 0$, $g(b) = 0 \Rightarrow$ By MVT, $\exists\, x_1 \in (0,b)$ s.t. $g'(x_1) = 0$

$\Rightarrow f(x_1)\cdot f''(x_1) + [f'(x_1)]^2 = 0$. ①

$f(1) > 0$, $f(0) = 0 \Rightarrow$ By MVT, $\exists\, x_2 \in (0,1)$, $f'(x_2) = \frac{f(1)-f(0)}{1-0} > 0$. Since $f'(0) < 0$,

there exists $x_3 \in (0, x_2)$ such that $f'(x_3) = 0$ by Intermediate Value Thm.

$\Rightarrow g(x_3) = f(x_3)\cdot f'(x_3) = 0 \Rightarrow$ By MVT, $\exists\, x_4 \in (b, x_3)$, $g'(x_4) = 0$

$\Rightarrow f(x_4)\cdot f''(x_4) + [f'(x_4)]^2 = 0$ ②

Above all, the equation $f(x)\cdot f''(x) + [f'(x)]^2 = 0$ has at least two solutions x_1 and x_4 in $(0,1)$.

#4.

$x^3+y^3-3x+3y-2=0$

differentiate $\Longrightarrow$ $3x^2+3y^2\cdot y'-3+3y'=0$

$\Rightarrow y' = \dfrac{3-3x^2}{3y^2+3} = \dfrac{1-x^2}{y^2+1} = 0 \Longrightarrow x=\pm 1.$ (two extreme pts).
$\Rightarrow y=1, 0$ respectively.

$\Rightarrow$ continue to differentiate: $6x+6y(y')^2+3y^2\cdot y''+3y''=0$

$\Rightarrow (3y^2+3)y'' = -6x-6y\cdot(y')^2$

Note: the sign chart of y' also shows the extreme value pts.

$\Rightarrow y'' = \dfrac{-6x-6y\cdot(y')^2}{3y^2+3} = \dfrac{-2x-2y\cdot(y')^2}{y^2+1}$

when $\begin{matrix} x=1 \\ y=1 \end{matrix}$: $y'=0$ and $y''=\dfrac{-2-0}{1+1}=-1<0$.

By the second derivative test, y gets maximum at $(1,1)$.

when $\begin{matrix} x=-1 \\ y=0 \end{matrix}$: $y'=0$ and $y''=\dfrac{2-0}{0+1}=2>0$

By the second derivative test, y gets minimum at $(-1,0)$.

#5.

Assume $P(a,b)$ is on L. Then $T: y-b=f'(a)(x-a)$,

$N: y-b=-\dfrac{1}{f'(a)}(x-a)$.

$\Rightarrow$ y-intercept of T is $b-a\cdot f'(a)$; x-intercept of N is $a+bf'(a)$.

$\Rightarrow b-a\cdot f'(a) = a+b\cdot f'(a) \Rightarrow f'(a)=\dfrac{b-a}{b+a}$ for any $P(a,b)$ on L.

$\Rightarrow L: f'(x)=\dfrac{y-x}{y+x}$ or $y'=\dfrac{y-x}{y+x} \Longrightarrow y'=\dfrac{\frac{y}{x}-1}{\frac{y}{x}+1}$. Let $\dfrac{y}{x}=u$

$\Rightarrow y'=u+xu'=\dfrac{u-1}{u+1} \Rightarrow \dfrac{1}{x}dx=-\dfrac{u+1}{u^2+1}du \Rightarrow \int\dfrac{1}{x}dx=\int-\dfrac{u+1}{u^2+1}du$

$\Rightarrow \ln(u^2+1)+2\arctan u=-2\ln|x|+C$. By $y(1)=0 \Rightarrow \ln(x^2+y^2)+2\arctan\dfrac{y}{x}=0$

#6. $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2} \ln(1+\frac{k}{n})$

$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n} \ln(1+\frac{k}{n}) \cdot \frac{1}{n} = \int_0^1 x \ln(1+x)\, dx$

$= \int_0^1 \ln(1+x)\, d(\frac{1}{2}x^2) = \ln(1+x)\cdot \frac{1}{2}x^2 \Big|_0^1 - \int_0^1 \frac{1}{2}x^2 \cdot \frac{1}{1+x}\, dx$

$= \ln 2 \cdot \frac{1}{2} - \frac{1}{2}\int_0^1 \frac{x^2}{1+x}\, dx$

$= \frac{\ln 2}{2} - \frac{1}{2}\int_0^1 x-1 + \frac{1}{1+x}\, dx$

$= \frac{\ln 2}{2} - \frac{1}{2}\left(\frac{1}{2}x^2 - x + \ln|1+x|\right)\Big|_0^1$

$= \frac{\ln 2}{2} - \frac{1}{2}\left(\frac{1}{2} - 1 + \ln 2\right) = \boxed{\frac{1}{4}}$

#7. $f(x) = \frac{1}{\ln(1+x)} - \frac{1}{x} - k \quad , \quad x \in (0,1)$.

$f(1) = \frac{1}{\ln 2} - 1 - k \quad , \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left[\frac{1}{\ln(1+x)} - \frac{1}{x} - k\right]$

$= \lim_{x\to 0^+} \frac{x - \ln(1+x)}{x\ln(1+x)} - k \overset{L'H}{=\!=} \lim_{x\to 0^+} \frac{1 - \frac{1}{1+x}}{\ln(1+x) + \frac{x}{1+x}} - k$

$= \lim_{x\to 0^+} \frac{(1+x) - 1}{(1+x)\ln(1+x) + x} - k = \lim_{x\to 0^+} \frac{x}{(1+x)\ln(1+x) + x} - k$

$\overset{L'H}{=\!=} \lim_{x\to 0^+} \frac{1}{\ln(1+x) + 1 + 1} - k = \frac{1}{2} - k$.

In order for $f(x) = 0$ to have a zero in $(0,1)$, we have $f(0)\cdot f(1) < 0$

$\Rightarrow \left(\frac{1}{\ln 2} - 1 - k\right)\left(\frac{1}{2} - k\right) < 0 \Rightarrow \left(k + 1 - \frac{1}{\ln 2}\right)\left(k - \frac{1}{2}\right) < 0$

$\Longrightarrow \boxed{\frac{1}{\ln 2} - 1 < k < \frac{1}{2}}$

#8

$$\lim_{t\to\infty}\int_0^t \frac{\ln(1+x)}{(1+x)^2}\,dx$$

$$=\lim_{t\to\infty}\int_0^t -\ln(1+x)\, d\frac{1}{1+x}$$

$$=\lim_{t\to\infty}\left[-\ln(1+x)\frac{1}{1+x}\Big|_0^t+\int_0^t\frac{1}{1+x}\cdot\frac{1}{1+x}\,dx\right]$$

$$=\lim_{t\to\infty}\left[-\frac{\ln(1+t)}{1+t}+\int_0^t\frac{1}{(1+x)^2}\,dx\right]$$

$$=\lim_{t\to\infty}\left[-\frac{\ln(1+t)}{1+t}-\left(\frac{1}{1+x}\right)\Big|_0^t\right]$$

$$=\lim_{t\to\infty}\left[-\frac{\ln(1+t)}{1+t}-\frac{1}{1+t}+1\right]$$

$$=\lim_{t\to\infty}\left[-\frac{\ln(1+t)+1}{1+t}+1\right]$$

$$\overset{L'H}{=}\lim_{t\to\infty}-\frac{\frac{1}{1+t}+0}{1}+1$$

$$=\lim_{t\to\infty}-\frac{1}{1+t}+1$$

$$=0+1$$

$$=\boxed{1}$$

#9

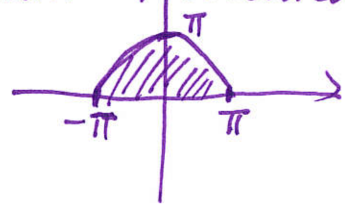
$$\int_{-\pi}^{\pi}\left(\sin^3x+\sqrt{\pi^2-x^2}\right)dx$$

$$=\int_{-\pi}^{\pi}\sin^3x\,dx+\int_{-\pi}^{\pi}\sqrt{\pi^2-x^2}\,dx$$

$$\overset{\text{the function}}{\underset{\sin^3x\text{ is odd}}{=}}0+\int_{-\pi}^{\pi}\sqrt{\pi^2-x^2}\,dx$$

$$=\int_{-\pi}^{\pi}\sqrt{\pi^2-x^2}\,dx$$

↳ area of the region under ~~the~~ semicircle $x^2+y^2=\pi^2$



$$=\frac{\pi\cdot\pi^2}{2}=\boxed{\frac{\pi^3}{2}}$$