Key - 2015

1. $\lim_{x\to 0} \frac{\ln \cos x}{x^2} \overset{L'H}{=} \lim_{x\to 0} \frac{\frac{1}{\cos x}\cdot(-\sin x)}{2x} = \lim_{x\to 0} \frac{1}{\cos x}\cdot\frac{(-1)}{2}\cdot\frac{\sin x}{x}$

$= 1\cdot\left(-\frac{1}{2}\right)\cdot 1 = \boxed{-\frac{1}{2}}$

2. $\lim_{x\to 0} \frac{x + a\ln(1+x) + bx\sin x}{kx^3} = 1$

$\overset{L'H}{\Rightarrow} \lim_{x\to 0} \frac{1 + \frac{a}{1+x} + b\sin x + bx\cos x}{3kx^2} = 1$

implication:

$\Rightarrow \lim_{x\to 0} 1 + \frac{a}{1+x} + b\sin x + bx\cos x = 0$

$\Rightarrow 1 + a = 0 \Rightarrow \boxed{a = -1}$

plug in

$\Rightarrow \lim_{x\to 0} \frac{1 - \frac{1}{1+x} + b\sin x + bx\cos x}{3kx^2} = 1$

implication:

$\lim_{x\to 0} (1+x)^{-2} + 2b\cos x - bx\sin x = 0$

$\Rightarrow 1 + 2b = 0$

$\Rightarrow \boxed{b = -\frac{1}{2}}$

$\overset{L'H}{\Rightarrow} \lim_{x\to 0} \frac{(1+x)^{-2} + b\cos x + b\cos x - bx\sin x}{6kx} = 1$

plug in

$\Rightarrow \lim_{x\to 0} \frac{(1+x)^{-2} - \cos x + \frac{1}{2}x\sin x}{6kx} = 1$

$\overset{L'H}{\Rightarrow} \lim_{x\to 0} \frac{-2(1+x)^{-3} + \sin x + \frac{1}{2}\sin x + \frac{1}{2}x\cos x}{6k} = 1 \Rightarrow \frac{-2+0}{6k} = 1$

$\Rightarrow \boxed{k = -\frac{1}{3}}$

3. $f(x) = \lim_{t\to 0}\left(1+\frac{\sin t}{x}\right)^{\frac{x^2}{t}}$ $\overset{1^\infty \text{ type}}{=} \lim_{t\to 0} e^{\ln\left(1+\frac{\sin t}{x}\right)^{\frac{x^2}{t}}} = e^{\lim_{t\to 0}\frac{x^2}{t}\ln\left(1+\frac{\sin t}{x}\right)}$

$= e^{\lim_{t\to 0}\frac{\ln\left(1+\frac{\sin t}{x}\right)}{\frac{t}{x^2}}} \overset{L'H}{=\!=} e^{\lim_{t\to 0}\frac{\frac{1}{1+\frac{\sin t}{x}}\cdot\frac{\cos t}{x}}{\frac{1}{x^2}}} = e^{\frac{\frac{1}{x}}{\frac{1}{x^2}}} = e^x$

So $f(x)$ is continuous on $(-\infty,\infty)$.

4. f' is continuous at $x=0 \Rightarrow f$ is continuous at $x=0$.

then $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0) = 0$ Note: $f(x) = x^\alpha \cos\frac{1}{x^\beta}$, $x>0$.

$\Rightarrow \lim_{x\to 0^+} x^\alpha \cos\frac{1}{x^\beta} = 0$. It works since $\alpha>0$.

$f'_+(0) = \lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} \alpha x^{\alpha-1}\cos\frac{1}{x^\beta} - x^\alpha \sin\frac{1}{x^\beta}\cdot(-\beta)x^{-\beta-1} = f'_-(0) = 0$.

$\Rightarrow \lim_{x\to 0^+} \alpha x^{\alpha-1}\cos\frac{1}{x^\beta} + \beta x^{\alpha-\beta-1}\sin\frac{1}{x^\beta} = 0$

$\Rightarrow \alpha-\beta-1>0 \Rightarrow \boxed{\alpha-\beta>1}$

5. $f(x) = x^2 2^x \Rightarrow f'(x) = 2x\cdot 2^x + x^2\cdot 2^x\ln 2 \Rightarrow f''(x) = 2\cdot 2^x + 2x\cdot 2^x\ln 2 + 2x\cdot 2^x\ln 2 + x^2 2^x(\ln 2)^2$

$= 2\cdot 2^x + [4\ln 2\cdot x + x^2(\ln 2)^2]\,2^x, \ldots$

$\Rightarrow f'(x) = (2x + \ln 2\cdot x^2)\cdot 2^x$

$f''(x) = [2 + 4\ln 2\cdot x + (\ln 2)^2\cdot x^2]\cdot 2^x$

$f'''(x) = [6\ln 2 + 6(\ln 2)^2 x + (\ln 2)^3\cdot x^2]\,2^x$

$f^{(4)}(x) = [12(\ln 2)^2 + 8(\ln 2)^3 x + (\ln 2)^4\cdot x^2]\cdot 2^x$

$f^{(5)}(x) = [20(\ln 2)^3 + 10(\ln 2)^4 x + (\ln 2)^5 x^2]\cdot 2^x$

$\vdots$

According to the pattern,

$f^{(n)}(x) = \left[(n-1)\cdot n(\ln 2)^{n-2} + 2n(\ln 2)^{n-1}x + (\ln 2)^n\cdot x^2\right]\cdot 2^x$

$\Rightarrow f^{(n)}(0) = \boxed{(n-1)\cdot n\cdot(\ln 2)^{n-2}}$

6. slope of the tangent line at $P = f'(b)$

$\parallel$ by the two points $(b, f(b))$, $(x_0, 0)$ $\Bigg\} \Rightarrow f'(b) = \dfrac{f(b)-0}{b-x_0}$

$\Rightarrow \boxed{f'(b) = \dfrac{f(b)}{b-x_0}}$.

$f(x) > 0 \Longrightarrow \dfrac{f(b)}{b-x_0} > 0$, and $f \nearrow \Rightarrow f(b) > f(a) = 0$

$\Longrightarrow b - x_0 > 0 \Rightarrow \boxed{x_0 < b}$

$\Rightarrow x_0 = b - \dfrac{f(b)}{f'(b)}$

Next, we need to prove $b - \dfrac{f(b)}{f'(b)} > a$, i.e. $x_0 > a$.

since $f'(b) > 0$, we only need to show $bf'(b) - f(b) > af'(b)$,

$\Longrightarrow$ assume $g(x) = xf'(x) - f(x) - af'(x) = (x-a)f'(x) - f(x)$.

then $g(a) = 0$, $g'(x) = f'(x) + (x-a)f''(x) - f'(x) = (x-a)f''(x)$

Note: $f'' > 0$. So $g'(x) > 0$ if $x > a \Rightarrow g(x) \nearrow$

$\Rightarrow g(b) > 0 \Longrightarrow bf'(b) - f(b) - af'(b) > 0$

$\Longrightarrow bf'(b) - f(b) > af'(b)$

$\Longrightarrow \boxed{x_0 > a}$

$\Longrightarrow \boxed{a < x_0 < b}$

7. $\phi(1)=1 \Rightarrow \int_0^1 f(t)\,dt = 1$, Note: $\phi(x)=\int_0^{x^2} x f(t)\,dt = x\cdot\int_0^{x^2} f(t)\,dt$

$$\phi'(x) = 1\cdot\int_0^{x^2} f(t)\,dt + x\cdot f(x^2)\cdot 2x$$

$$\Rightarrow \phi'(1) = \int_0^1 f(t)\,dt + 2f(1) = 5 \Rightarrow 1+2f(1) = 5$$

$$\Rightarrow \boxed{f(1)=2}$$

8. $f(x)=\int_0^x e^{-u}\cos u\,du \Rightarrow f'(x) = e^{-x}\cos x$, $x\in[0,\pi]$

$\Rightarrow$ critical #s in $[0,\pi]$: $\frac{\pi}{2}$

then $f(0)=0$, $f(\pi)=\int_0^{\pi} e^{-u}\cos u\,du \xrightarrow[\text{by parts}]{\text{integration}} \frac{1}{2}(1+e^{-\pi})$

$$f(\tfrac{\pi}{2}) = \int_0^{\frac{\pi}{2}} e^{-u}\cos u\,du = \frac{1}{2}(1+e^{-\frac{\pi}{2}})$$

$\Rightarrow \max(f) = f(\frac{\pi}{2}) = \boxed{\frac{1}{2}(1+e^{-\frac{\pi}{2}})}$, $\min(f) = f(0) = \boxed{0.}$

9. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin x}{1+\cos x}\,dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |x|\,dx$

$$= -\ln(1+\cos x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + 2\int_0^{\frac{\pi}{2}} x\,dx$$

$$= 0 + x^2\Big|_0^{\frac{\pi}{2}} = \boxed{\frac{\pi^2}{4}}$$

10. $\displaystyle\int \frac{dx}{(1+e^x)^2} = \int -e^{x} \, d\left(\frac{1}{1+e^x}\right)$ integration by parts

$$= \frac{-e^{-x}}{1+e^x} - \int \frac{1}{1+e^x}\, d(-e^{-x})$$

$$= \frac{-e^{-x}}{1+e^x} - \int \frac{e^{-x}}{1+e^x}\, dx$$

$$= \frac{-e^{-x}}{1+e^x} - \int \frac{1}{e^x(1+e^x)}\, dx$$

$$= \frac{-e^{-x}}{1+e^x} - \int \frac{1}{e^x} - \frac{1}{1+e^x}\, dx$$

$$= \frac{-e^{-x}}{1+e^x} - \int e^{-x}\, dx + \int \frac{1}{1+e^x}\, dx$$

$$= \frac{-e^{-x}}{1+e^x} + e^{-x} + \int \frac{1}{1+e^x}\, dx$$

$$= \frac{-e^{-x} + e^{-x}(1+e^x)}{1+e^x} + \int \frac{1+e^x - e^x}{1+e^x}\, dx$$

$$= \frac{-e^{-x} + e^{-x} + 1}{1+e^x} + \int \frac{1+e^x}{1+e^x} - \frac{e^x}{1+e^x}\, dx$$

$$= \frac{1}{1+e^x} + \int 1 - \frac{e^x}{1+e^x}\, dx$$

$$= \frac{1}{1+e^x} + \int 1\, dx - \int \frac{e^x}{1+e^x}\, dx$$

$$= \boxed{\frac{1}{1+e^x} + x - \ln(1+e^x) + C}$$