

The Total Variation Approach to Approximate Hyperbolic Wave Equations

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ABSTRACT

From ultrasound for medical purposes to exploration for oil drilling, waves are everywhere. Knowing how waves should move can help others to work faster and more accurately. The study of wave propagation is analyzing how waves move and how they react to certain obstacles.

When the exact function for a set of data that describes a wave is not known, an approximation is needed to move the wave. Dr. Robert Hoar of the Mathematics Department and I have worked on creating modified numerical methods to handle moving these waves without creating oscillations (the wiggles in the graph) using polynomials (simple functions) to approximate the motion of the wave. We used a complicated selection routine in the attempt to reduce these wiggles [1].

For this research, we worked on refining these numerical methods. We were trying to create computationally better method. We used a method that measured the size of the oscillations, called *total variation*, in our decision making routine.

THE EQUATION

We are studying a hyperbolic partial differential equation. This is a one way, one-dimensional partial differential equation. These waves follow this condition.

$$u_t + cu_x = 0 \quad \text{where } c \text{ is the wave speed (here } c=1) \text{ and}$$

$$u_t(x,t) = -u_x(x,t) \quad u \text{ is the displacement from equilibrium at position } (x,t)$$

One could imagine this wave as a wave on a thin wire or a pulse on along a garden hose. Using the Taylor's Expansion about the point (x,t) , $u(x, t + \Delta t)$ can be represented as:

$$u(x, t + \Delta t) = u(x,t) + u_x(x,t) \Delta t + (1/2!) u_{xx}(x,t) \Delta t^2 + \dots + (1/n!) u_{x^n}(x,t) \Delta t^n + R_{n+1}(x,t) \quad [2]$$

We can note that if the function is smooth, the Taylor's expansion will converge, and the $R_{n+1}(x,t)$ remainder go to zero. If the function is not as smooth, the later terms of the Taylor's expansion blow up, and the $R_{n+1}(x,t)$ remainder may not go to zero.

Since we know that the above condition holds for these wave equations, we can substitute space derivatives (those with respect to x) for the time derivatives.

$$u(x, t + \Delta t) = u(x,t) + u_x(x,t) \Delta t + (1/2!) u_{xx}(x,t) \Delta t^2 + \dots + (1/n!) u_{x^n}(x,t) \Delta t^n + R_{n+1}(x,t)$$

Now, we need to find how to approximate the space derivatives.

Finite Differences

We do this using a numerical method called *finite difference approximations*. Again, we look at Taylor's Expansions.

$$f(x) = f(x)$$

$$f(x + \Delta x) = f(x) + f'(x) \Delta x + \frac{f''(x)}{2!} \Delta x^2 + \dots$$

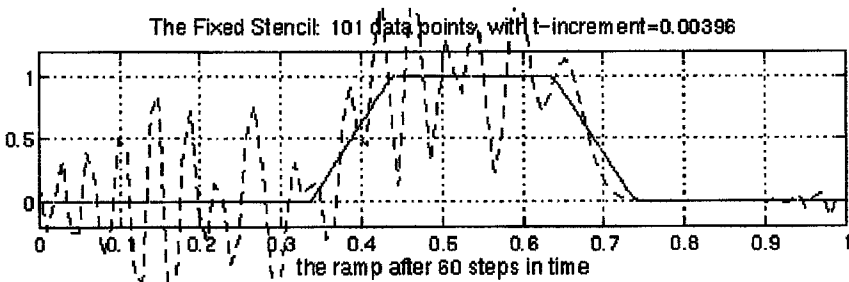
We can create a linear combination of these values to determine the derivative at $f(x)$ or $f'(x)$.

$$f'(x) = \frac{f(x + \Delta x) - f(x)}{\Delta x} + O(\Delta x) \quad (\text{See [2] for } O \text{ notation.})$$

This is called a \bullet \circ stencil with the dot representing the point we are finding the derivative at and the circle representing the extra points to calculate the derivative. We would use this to “step” forward in time using:

$$u(x, t + \Delta t) = u(x, t) - \Delta t (f'(x) \text{ approximate}) + O(\Delta t^2)$$

Our original problem was that we wanted to find a “good” numerical solution to move this wave with no oscillations and with high resolution. Standard methods of determining approximations can introduce these oscillations into the approximation [3]. This illustrates how far the fixed stencil method (the dashed line) can stray from the true solution (the solid line). Even after only 60 steps in time, the error is clear.



So, we decided to change the points used in the finite difference, or stencil, depending on the data. This stencil can include one to as many data points as given. Using more data points is more accurate, but it takes more work to compute. Using different stencils on the data in difference case can also improve accuracy.

The Definition of Total Variation

Total variation, a mathematical technique, measures how much a function moves up and down. If it has too many oscillations, the total variation is relatively higher.

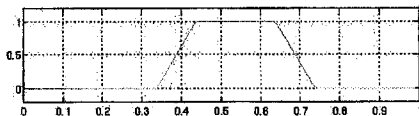


figure A

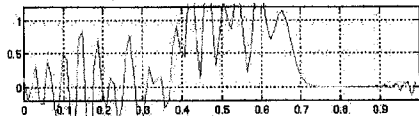


figure B

For example, figure A has a much smaller Total Variation than figure B, since figure B has more ups and downs.

The Definition of *ENO*

We know oscillatory functions have large total variations. We used a scheme called *Essentially Non-Oscillatory*, or *ENO*, [4]. *ENO* schemes seek to reduce the oscillations by limiting the growth of the total variation. Fixed stencil finite difference methods are not *ENO*.

The *ENO* method involves creating a sequence of interpolating polynomials and using a complicated selection routine to pick the correct stencil. We used this method to gauge how well the new methods were doing. We worked on other variations of this idea involving more terms of the Taylor's Expansion.

OUR RESEARCH

For this research, we realized that we were trying to limit the growth of the total variation of the approximation. It seems natural to limit the total variation in the selection routine. Our new methods implement this.

We developed three different methods to use the total variation in the selection routine. We studied a second order stencil. That is a finite difference using three data points. All these numerical methods use the polynomial interpolant to make decisions on which stencil to use if it has a choice. Since we have a positive wave speed (c), we need to look at the $\bigcirc\bigcirc\bullet$ stencil and the $\bigcirc\bullet\bigcirc$ stencil. For this analysis, let the polynomial interpolant for the $\bigcirc\bigcirc\bullet$ stencil be P_1 . Let the polynomial interpolant for the $\bigcirc\bullet\bigcirc$ stencil be P_2 .

Method #1: Multiple-Point Total Variation Method

This method takes the total variation of the polynomial interpolants over all 3 points for each stencil. Taking the total variation, or TV, over the 3 points included in the stencil, we would pick the $\bigcirc\bigcirc\bullet$ stencil if $TV(P_1) \leq TV(P_2)$. This will give us an idea if the interpolant is relatively smooth in this region or not. We conjectured that a smooth polynomial interpolant over these three points would imply that the stencil selection would be more stable over time. Here, more stable means it will not introduce the oscillations or if it does it will do so at a slower rate.

Method #2: Two-Point Total Variation Method

In looking for a computationally cheaper method, we came upon this idea. This method takes the total variation of the polynomial interpolants over the two downwind points for each stencil. So, we compute the total variation over just the $\bigcirc\bullet$ part of the stencil. Taking the total variation, or TV, over those 2 points, $\bigcirc\bullet$, for the $\bigcirc\bigcirc\bullet$ stencil and for the $\bigcirc\bullet\bigcirc$ stencil, we would pick the $\bigcirc\bigcirc\bullet$ stencil if $TV(P_1) \leq TV(P_2)$. This will give us an idea if the interpolant is relatively smooth in this region or not. We conjectured that a smooth polynomial interpolant near the point we were calculating the derivative at would imply that the stencil selection would be more stable over time.

Method #3: Divided Total Variation Method

Some of the data sets we were testing these methods on have some extremely large jumps. This is quite a shock for the standard methods. We were concerned that the selection routine might be choosing the incorrect stencil. This method takes the

total variation of the polynomial interpolants over all 3 points for each stencil as in the Multi-Point Total Variation Method, then it divides this by the total variation of the data itself. Taking the total variation, or TV, over the 3 points included in the stencil, we would pick the (○○●) stencil if $\frac{TV(P1)}{TV(○○●)} \leq \frac{TV(P2)}{TV(○●○)}$

We hoped that this would indicate if the interpolant is relatively smooth in this region or not taking into account the region the total variation was measured over. Again, we conjectured that a smooth polynomial interpolant after the division near the point we were calculating the derivative at would imply that the stencil selection would be more stable over time.

INTRODUCTION OF THE ANALYSIS

The purpose of this research was to determine if this new method is an improvement over the other methods. Before we can say if a method is better than another, we have to define what “better” is. A method is “better” if it uses less computer operations than the other methods and maintains the same level of accuracy. The method can also be considered “better” if it uses the same number of computer operations but is more accurate.

We did most of this analysis on the methods using a computer and Matlab, a mathematics program. I wrote the programs to implement these methods and other programs to test these methods. We generated graphs of the results for visual inspection. We also calculated the error at each stage. From these sources, we could identify how well a method was working. From this analysis, we can judge if the oscillations are reducing or becoming larger.

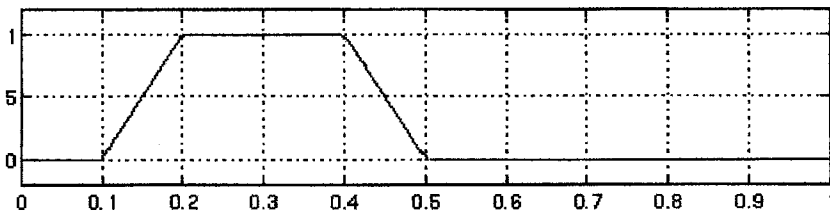
We have a couple different criteria for judging how well the method does. What we learned from the visual cues from the graphs is also evident when the norms of the error are examined. We want to know how much distance (or area) is between the original and our result. That is the *error*. Also, we analyze how much work (computer computations) is necessary to carry out this process. That is the *computational complexity*.

ANALYSIS

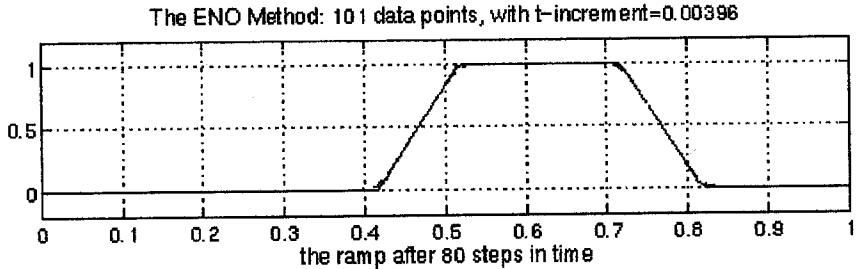
We had eight data sets to test the method against. Each data set consists of discrete data. We used a Δx of 0.01 and a Δt of 0.004. For each data set that we would test, we would make a copy of the original data for each of the methods. We would let our method compute the next step in time. Then we would determine how well each does. For reference, we compared our methods to *ENO* as our standard method.

Here, we are going to examine the “ramp” data set.

The Function: 101 data points, with t-increment=0.00396



This is a very simple data set; it is four lines connected together. Where the lines are connected, the function is continuous, but it is not differentiable. This effects the Taylor expansion of the wave. Also, we are using periodic boundary conditions. So once the wave gets to the right side of the box, it starts scrolling onto the left side of the box.

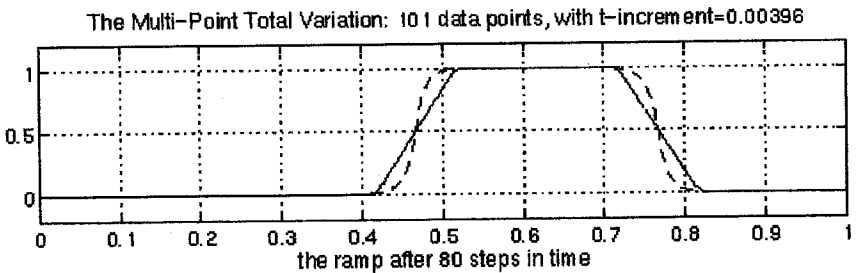


Let's look at the how well the ENO method does on this data set after 80 steps in time. (The dashed is the approximate. The solid line is the true solution.)

This method is very stable (or does not introduce oscillations at this stage).

The following are the characteristics of this method.

- The result of this method captures the shape of the wave at each time step.
- The result of this method is almost exact everywhere except on the "corners" where the lines come together at a junction.
- This method has a relatively small error (see Table 1).
- The method is more computationally complex than the other methods.
- The costs are worth the results.



Let's look at the how well the Multiple-Point Total Variation method does on this data set after 80 steps in time. (The dashed is the approximate. The solid line is the true solution.)

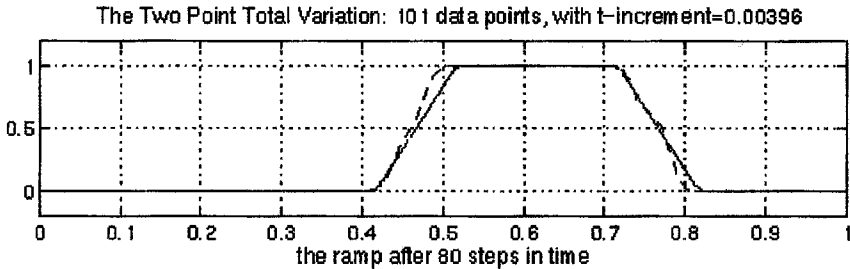
This method is not as stable as the *ENO* method.

The following are the characteristics of this method.

- The result of this method is starting to not hold on to the shape of the wave at each time step.

- The result of this method is trying to turn this ramp into a step function where it would jump from 0 to 1 and back to 0.
- It should be capturing the sloping sides better.
- The error of this method is not large (see Table 1).
- The method is less computationally complex than the *ENO* method.
- The results on this function are okay considering the computational costs.

Let's see if the Two-Point Total Variation method does any better. This is also on this data set after 80 steps in time. (The dashed is the approximate. The solid line is the true solution.)

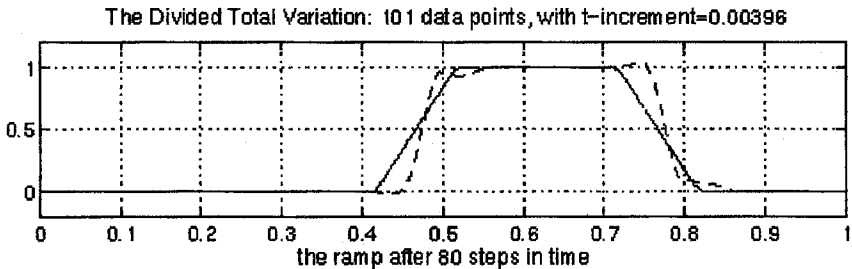


This method is not as stable as the *ENO* method.

The following are the characteristics of this method.

- The result of this method usually captures the shape of the wave at each time step.
- The result of this method should not weave from one side of the wave to the other on the sloping sides of the ramp.
- The method has a relatively small error (see Table 1).
- The method is the least computationally complex of all these methods.
- The results are very good considering the computational costs.

Lastly, let's look at the Divided Total Variation method after 80 time steps on this data set. (The dashed is the approximate. The solid line is the true solution.)



This method is not stable.

The following are the characteristics of this method.

- The result of this method does not capture the shape of the wave at each time step.
- The result of this method on the "corners" causes a good deal of error.
- It should capture the sloping sides better.
- The error of this method is relatively large on most data sets.

- The error of this method is larger than the others (see Table 1).
- The method is less computationally complex than the ENO method, but it is more computationally complex than the other new methods.
- The results on this function are awful considering the computational costs.

These new methods have many of the positive attributes of other methods but with less error than many others. Let's look at how we compute the error. We take the norm of the vector that held the values from the method minus the values from the true solution. Since we are subtracting the new one from the true solution, we want the numbers to be closer to *zero*. Some of the errors are in Table 1 below.

Table 1

	Initially	Time = 40	Time = 60	Time = 80
<i>ENO</i>	0.0	0.0614	0.0661	0.06907
Multi-Point TV	0.0	0.4670	0.6324	0.72140
Two Point TV	0.0	0.2696	0.3850	0.42630
Divided TV	0.0	0.4348	0.6842	0.90970

While certain computational issues remain to be studied, it appears that these methods will be comparable in computation complexity to the standard methods. In fact, the new methods may be slightly less complex than methods of comparable accuracy.

CONCLUSION

We want to predict how waves move. This is important, because waves are everywhere. Knowing how waves move can help people to optimize time and productivity when working with waves. We studied these methods in order to help us to predict the movement of waves.

One technique used to measure how much a function moves up and down is called "total variation." If an approximation has too many oscillations (the wiggles in the graph), the total variation is larger. Reducing the total variation can bring approximations closer to the true solution. That is what we want. While these total variation methods are not as stable as the original *ENO* method, with some more work, these methods could be refined. Since these methods are less computationally expensive than the standard methods, this would be a great asset to have such a method.

REFERENCES

- [1] Robert H. Hoar, Ph.D. *An Adaptive Stencil Finite Difference Method for First Order Linear Hyperbolic System*, 1995.
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- [4] A. Harten and S. Osher. *Uniformly High-Order Accurate Nonoscillatory Schemes I*. SIAM Journal on Numerical Analysis, 24(2):279-309, 1987.