

Optimal Travel Path

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ABSTRACT

When determining the course of a satellite, one usually has some key “targets” or cities that the satellite would be directed over. This paper deals with finding a means by which one could determine the optimal path of a given shape for any given set of cities (data). Here, optimal means that some measurement of distance between the set of cities/targets and the path of the satellite has been minimized. First, an appropriate method that could be used to determine each distance was determined. Finally, mathematics software was used to create a program which would be able to approximate the minimum of the sum of all calculated distances.

INTRODUCTION

For large numbers of the cities/targets, it is impossible to pass directly over all cities/targets in a single pass of a satellite. Suppose the path of the satellite travels over a given region is parabolic in form (the mathematical formula for such a path is $f(x) = ax^2 + bx + c$, where a , b , and c are constants that determine the shape of the parabola). The question considered is the following: How does one find the best (optimal) path in order to come closest to a list of cities? For a parabolic path, this problem requires the determination of the optimal values of a , b , and c . The problem of finding an optimal path will require the minimization of a function of more than one variable. By studying the least squares approach, insight was gained into the method by which the minimization problem can be solved. The theory was then extended so as to apply to a new minimization problem. Finally, the two new methods were implemented on random data sets and the results were compared and contrasted.

The expected outcome of this endeavor was to determine a means by which one can determine the optimal path of a given shape for any given set of cities (data). To do this, a method of measurement that would minimize sum was selected (there is more than one way to measure distance). The next section contains a description of one such method. The method is an extension of the Least Squares method (least squares does not yield an acceptable path).

METHOD

The simplest path for the satellite is linear, but this does not provide sufficient control. The linear model was extended to include a degree two term arriving at a model of the form:

$$f(x) = ax^2 + bx + c$$

So, we must find the optimal values of a , b , and c , which would result in the optimal path, i.e. the sum of the distances to be minimized.

While expecting to solve the problem with any number of cities, starting with very few cities simplifies the problem. It can be shown that there is a parabolic path that can pass directly over any three points on a map. The interesting problem arises when a fourth city is

added to the map. In general, there is no single parabolic path that directly goes over four arbitrary cities. Since the path cannot pass directly above all four cities, a path must be found that comes closest to this goal. Choosing an interpolating parabola (one that passes over three of the cities exactly) leaves one city/target not directly on the path. The distance between the point, say at (x_4, y_4) , and the path of the satellite, $f(x) = ax^2 + bx + c$, is given by $[(x-x_4)^2 + (y-y_4)^2]^{1/2}$ (see[3]).

The obstacle was to decide on a method that would help determine the minimal sum of distances if a, b , and c are allowed to be any value. There were two methods considered: 1) The Least Squares Method and 2) The Minimization of the Perpendicular Distance Method. In the Least Squares method, a, b , and c are chosen so that the parabolic path minimizes the sum of the squares of the vertical distances from the point/targets to the path of the satellite. In Figure 1, the Least Squares Optimal solution to a data set is shown.

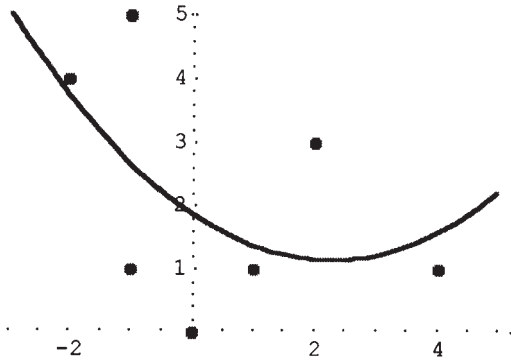


Figure 1: Least Squares solution to the Optimal Path problem.

The Least Squares Method minimizes the sum of the squares of the north-south distances. For an optimal path, the more reasonable distance to consider is the distance in the plane (any direction) from each city to the path, not just the north-south direction. The Least Squares solution also depends on the orientation of the map. If one were to turn the map a small amount, redraw a vertical and horizontal axis, and recalculate the least squares fit, one would arrive at a different optimal solution in the Least Squares sense (which implies the method is not selecting a truly optimal path).

When minimizing the perpendicular distance, the main idea is to minimize the sum of the perpendicular distance from each city/target to the tangent line of the path of the satellite. Substituting in the path $f(x)$ of the satellite for y , into the distance equation, the distance for the fourth point becomes (see [3])

$$d = [(x-x_4)^2 + (ax^2 + bx + c - y_4)^2]^{1/2}$$

To minimize this distance (which is a function of x), the derivative of this equation must be set to zero, yielding

$$2a^2x^3 + 3abx^2 + (1 + 2ac - 2ay_4 + b^2)x + bc - by_4 - x_4 = 0.$$

Note that in this calculation, the values of a, b, c, x_4 and y_4 are all known.

The difference between Least Squares and Perpendicular Distance is easily seen with an example. In Figure 2, a parabola which interpolates 3 out of 4 points is presented. Note that the actual distance in the plane from the parabola to the fourth point is much smaller than the vertical distance from the point North to the parabola.

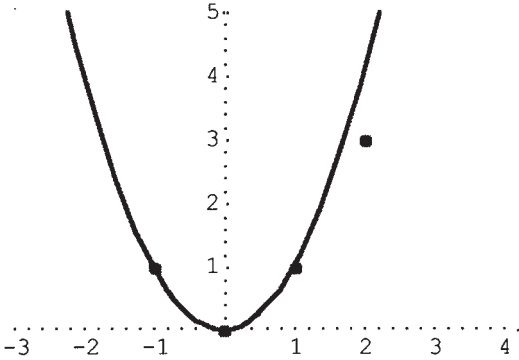


Figure 2: A parabola which interpolates 3 out of 4 points.

The problem of minimizing the sum of the perpendicular distances does not suffer the rotational affects that hinder the Least Squares approach. If the data can be reprinted as a collection of points on a grid, the orientation of the grid does not affect the distance from the data points to the curve, with the exception of a few degenerate cases.

In order to calculate individual distances, the sum of the perpendicular distance from the tangent line of the path of the satellite to the city/target (given n cities) should be minimized.

In other words,

$$\sum_{i=1}^n d(f, (x_i, y_i))$$

should be minimized over all possible values of a, b, and c. Here, d is the distance between f(x), the parabolic function, and (x_i, y_i), the ith city/target.

As mentioned above, in order to find a single distance, the derivative of the distance function needs to be zero, which requires the solution of the degree three problem

$$2a^2x^3 + 3abx^2 + (1 + 2ac - 2ay_4 + b^2)x + bc - by_4 - x_4 = 0.$$

Solving this results in three roots X₁, X₂, and X₃ given by following expressions. The distance d is taken to be the value that is smallest when x is taken to be the three following values. The value I is the imaginary number (-1)^{i/2}.

$$\begin{aligned}
X_1 &= (-b/(2a)-(6(a^2)-3(a^2)(b^2)+12(a^3)c- \\
&12(a^3)y_4)/(32^{2/3}(a^2)(54(a^3)b+108(a^4)x_4+(54(a^3)b+ \\
&108(a^4)x_4)^2+4(6(a^2)-3(a^2)(b^2)+12(a^3)c-12(a^3)y_4)^3)^{1/2})+ \\
&1/(62^{1/3}(a^2)((54(a^3)b+108(a^4)x_4+(54(a^3)b+108(a^4)x_4)^2+ \\
&4(6(a^2)-3(a^2)(b^2)+12(a^3)c-12(a^3)y_4)^3)^{1/2})^{1/3})) \\
X_2 &= -b/(2a)+((1+I3^{1/2})(6(a^2)-3(a^2)(b^2)+12(a^3)c- \\
&12(a^3)y_4)/(62^{2/3}(a^2)(54(a^3)b+108(a^4)x_4+(4(6(a^2)- \\
&3(a^2)(b^2)+12(a^3)c-12(a^3)y_4)^3+(-54(a^3)(b^3)+54(a^3)b(1+ \\
&(b^2)+2ac-2ay_4)-108(a^4)(bc-x_4-by_4)^2)^{1/2})^{1/3})- \\
&(1/(122^{1/3}a^2))((1-I3^{1/2})(54(a^3)b+108(a^4)x_4+(4(6(a^2)- \\
&3(a^2)(b^2)+12(a^3)c-12(a^3)y_4)^3+(-54(a^3)(b^3)+54(a^3)b(1+(b^2)+ \\
&2ac-2ay_4)-108(a^4)(bc-x_4-by_4)^2)^{1/2})^{1/3})) \\
X_3 &= -b/(2a)+((1-I3^{1/2})(6(a^2)-3(a^2)(b^2)+12(a^3)c- \\
&12(a^3)y_4)/(62^{2/3}(a^2)(54(a^3)b+108(a^4)x_4+(4(6(a^2)- \\
&3(a^2)(b^2)+12(a^3)c-12(a^3)y_4)^3+(-54(a^3)(b^3)+54(a^3)b(1+ \\
&(b^2)+2ac-2ay_4)-108(a^4)(bc-x_4-by_4)^2)^{1/2})^{1/3})- \\
&(1/(122^{1/3}a^2))((1+I3^{1/2})(54(a^3)b+108(a^4)x_4+ \\
&4(6(a^2)-3(a^2)(b^2)+12(a^3)c-12(a^3)y_4)^3+ \\
&(-54(a^3)(b^3)+54(a^3)b(1+(b^2)+2ac-2ay_4)- \\
&108(a^4)(bc-x_4-by_4)^2)^{1/2})^{1/3}))
\end{aligned}$$

With the aid of Mathematica [1] and Matlab [2], exact distances for individual points for parabolic satellite flights have been found. In addition, numerical approximations have been developed to find the minimum of the sum of all distances. Matlab's built in *fmins*, function which uses the Steepest Decent Method, was used. The *fmins* routine finds relative minima, not necessarily the absolute minimum. The Steepest Decent Method finds the local minimizer of a function that is nearest to a given set of initial values. So in order to find absolute minima, the *fmins* function needs a "near-by" initial value. The sum of the distances will have 3 terms which are zero if the coefficients are chosen to be those of an interpolating parabola, so the coefficients of all parabolas that interpolate three of the targets were selected as initial values in the *fmins* program. The *fmins* program was then extended so that the minimum of the local minima was reported.

LIMITATIONS

The limitation encountered revolved around the definition of rotational invariance for parabolic paths that are not defined as a function of x . It may be that the optimal path, as defined by the proposed minimization criteria, may not be a function of x . If the optimal path were a path which could double back on itself (like the function $x=y^2$ for example), then the modified *fmins* routine would not arrive at this path. If one assumed that the satellite *must* pass from the left to the right over the cities (or right to left), then the satellite would not be allowed to double back and the approach used above would no longer have such a limitation

CONCLUSION

An approach for finding individual distances was developed and used to extend the *fmins* function of Matlab to find optimal travel paths for satellites. The criteria used for defining the term optimal was selected under reasonable assumptions.

Though many results have been found, there are still many areas to research. Further research could be conducted in this area by changing the path of the parabola from a degree two to a degree n polynomial, or even changing from a polynomial to a more arbitrary function. Also considered could be a third measurement of distance, which would work by minimizing the maximum perpendicular distance from the cities/targets to the path of the satellite.

ACKNOWLEDGMENTS

I want to thank the University of Wisconsin-La Crosse Undergraduate Research Program for funding the research. I would also like to take the time to thank Dr. Hoar for his guidance and expertise throughout the research.

REFERENCES

- 1 Mathematica is product of Wolfram Research Inc, Champaign, IL.
- 2 Matlab is a product of MathWorks Inc, Natick, MA.
- 3 Thomas, George B. and Ross L. Finney. *Calculus and Analytic Geometry* 9th edition. Reading, Massachusetts: Addison-Wesley Publishing Company, June, 1996, p 17.